On Random Vandermonde Matrices

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Random Vandermonde Matrices

Let $V_n$ be the $n \times m$ random matrix of the form

$$V_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & \cdots & 1 \\ e^{2\pi i \theta_1} & \cdots & e^{2\pi i \theta_m} \\ \vdots & \ddots & \vdots \\ e^{2\pi i (n-1) \theta_1} & \cdots & e^{2\pi i (n-1) \theta_m} \end{bmatrix}$$

where the $\{\theta_1, \ldots, \theta_m\}$ are random variables in $[0, 1]$. 
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We are interested in the limit eigenvalue distribution of the random matrix $V_n^* V_n$ under the conditions that:

- The phases are i.i.d. in the interval $[0, 1]$ with prob. dist. $\nu$

- $\lim_{n \to \infty} \frac{n}{m} = c \in (0, \infty)$
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Previous work from [Nordio, Chiasserini, Viterbo, Ryan, Debbah and Norberg]
More generally, we study the random matrices $V_n^{(d)}$ for $d \geq 1$

Let $\ell = (\ell_1, \ldots, \ell_d) \in \{0, 1, \ldots, n - 1\}^d$ and consider the function

$$f(\ell) = \sum_{j=1}^{d} n^{j-1} \ell_j$$

a bijection to the set $\{0, 1, \ldots, n^d - 1\}$

Consider $x_1, \ldots, x_m$ random vectors in $[0, 1]^d$ independent and identically distributed.

We know define the matrix

$$V_{f(\ell), q} = n^{-\frac{d}{2}} \exp(2\pi i \langle \ell, x_q \rangle)$$

We are interested in the limit eigenvalue distribution of the random matrix $(V_n^{(d)})^*(V_n^{(d)})$ under the condition $\lim_{n \to \infty} \frac{n^d}{m} = c \in (0, \infty)$
Why do we care?

Let \( m \) wireless sensors measuring the value of a spatially finite physical field (air temperature, pressure, etc) defined over a \( d \) dimensional compact space.

- One can think of sensor nodes randomly deployed over the geographical region.
- We want to reconstruct the field from a collection of samples that are noisy.
- Reconstruction base on the DFT.
- Measure reconstruction accuracy by the MSE.
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We can obtain the MSE as

$$MSE^{(m)} = \text{snr}^{-1} \frac{n^d}{m} \cdot \mathbb{E} \left( \text{tr} \left\{(V_n^{(d)})(V_n^{(d)})^* + \text{snr}^{-1} \frac{n^d}{m}\right\} \right)$$

The bottom line is that we need to understand the spectrum of $(V_n^{(d)})^*(V_n^{(d)})$
Some Related Polytopes

Let $\rho$ be the partition $\rho \in \mathcal{P}(k)$

$$\rho = \{B_1, \ldots, B_{|\rho|}\}$$

with $|\rho|$ blocks.
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For each \( i \in \{1, 2, \ldots, k\} \) consider a variable \( 0 \leq x_i \leq 1 \) and for each block \( j \) consider the equation

\[ E_j : \sum_{i \in B_j} x_{i-1} = \sum_{i \in B_j} x_i \]
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The solution set has $k + 1 - |\rho|$ free variables. We define

$$K_\rho = \text{volume of the solution set in } [0, 1]^{k+1-|\rho|}$$
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Example

Let $\rho = \{\{1, 3\}, \{2\}, \{4\}\}$. The number of free variables is 2

$$E_1 : x_1 + x_3 = x_2 + x_4 \quad \& \quad E_2 : x_2 = x_1 \quad \& \quad E_3 : x_4 = x_3$$

then $x_1 = x_2$ and $x_3 = x_4$ therefore

$$K_\rho = 1$$
Example

Let $\rho$ be $\{\{1, 3\}, \{2, 4\}\}$, the number of free variables is 3 and

$$E_1 = E_2 : x_1 + x_3 = x_2 + x_4$$

then

$$K_\rho = \text{vol}\left(\{(x_1, x_2, x_3) \in [0, 1]^3 : 0 \leq x_1 + x_3 - x_2 \leq 1\}\right) = \frac{2}{3}$$

**Figure**: Corresponding Polytope
Moments’ formulas

**Theorem (Ryan and Debbah)**

Assume that \( \nu \) has a continuous density \( p(x) \) on the unit interval. Then the asymptotic \( k \)-th moment of \( V_n^* V_n \) exists and is equal to

\[
m_k = \lim_{m \to \infty} \mathbb{E} \left[ \text{tr}_m \left( V_n^* V_n \right)^k \right] = \sum_{\rho \in P(k)} K_{\rho, \nu} c^{\lvert \rho \rvert - 1}
\]

where \( c = \lim \frac{n}{m} \) and

\[
K_{\rho, \nu} = K_{\rho} \cdot \int_0^1 p(x)^{\lvert \rho \rvert} \, dx
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and \( \lvert \rho \rvert \) is the number of blocks of \( \rho \).
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- For every partition $\rho$ we have $0 < K_{\rho} \leq 1$ and corresponds to the uniform distribution.
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- For every partition $\rho$ we have $0 < K_\rho \leq 1$ and corresponds to the uniform distribution
- $K_\rho$ is a rational number for every $\rho \in \mathcal{P}(k)$
- $K_\rho = 1$ if and only if $\rho$ is non–crossing
$d$-fold case

Random Vandermonde matrix defined as before with phase distribution $\nu$ such that:

Theorem (T., Whiting)

For every $d \geq 1$ we have that:

$$m(d)_{\nu, k} = \operatorname{lim}_{n \to \infty} E \operatorname{tr} m_n^* (V(d)_{\nu})^k$$

Moreover, there exists a unique limit measure $\mu(d)_{\nu, c}$ with these moments and it has unbounded support for every $d$.

Note that for the uniform distribution on $[0, 1]$:

$$m(d)_{k} = \sum_{\rho \in \mathcal{P}(k)} K_d \rho, \nu c = \sum_{\rho \in \mathcal{NC}(k)} c |\rho|^{-1} \to d \to \infty$$

which is the $k$-th moment of the Marchenko-Pastur distribution!
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For every $d \geq 1$ we have that

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Moreover, there exists a unique limit measure $\mu_{\nu, c}^{(d)}$ with these moments and it has unbounded support for every $d$.

Note that for the uniform distribution on $[0, 1]^d$

$$m_k^{(d)} = \sum_{\rho \in \mathcal{P}(k)} K_{\rho}^d c^{\rho|\rho|^{-1}} \xrightarrow{d \to \infty} \sum_{\rho \in \mathcal{NC}(k)} c^{\rho|\rho|^{-1}}$$

which is the $k$–th moment of the Marchenko-Pastur distribution!
Maximum eigenvalue

Let $V$ be a square $n \times n$ random Vandermonde matrix with phase distribution $\nu$ with density $p$

$$\{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\}$$

the e-values of $V^*V$

Theorem (T., Whiting)

$$\log n \log \log n (1 - o(1)) \leq E(\lambda_n) \leq \frac{4\pi}{\|p\|_{\infty}} (e^{-1} + 1) \log n + o(1)$$

We have similar results for the $d$-fold case.
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What is the behaviour of $\mathbb{E}[\lambda_n]$ as a function of $n$?
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What is the behaviour of \( \mathbb{E}[\lambda_n] \) as a function of \( n \)?

**Theorem (T., Whiting)**

\[
\frac{\log n}{\log \log n}(1 - o(1)) \leq \mathbb{E}(\lambda_n) \leq \left(4\pi \|p\|_{\infty}(e - 1) + 1\right) \log n + o(1)
\]

We have similar results for the \( d \)-fold case.
Minimum Eigenvalue

Given a vector $x$ in $\mathbb{C}^n$, we define $\sigma_r^m(x)$ to be the sum of all $r$-fold products of the components of $x$ not involving the $m$–th coordinate. In other words,

$$\sigma_r^m = \sum_{\rho_r^m} \prod_{k \in \rho_r^m} x_k$$

where $\rho_r^m$ is a subset of \{${x_1, x_2, \ldots, x_{m-1}, x_{m+1}, \ldots, x_n}$\} with cardinality $r$. 

Theorem (Macon and Spitzbart)

Let $V$ be a square $n \times n$ matrix given by

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 \\ \end{pmatrix}$$

with no 0 entries. Then its inverse $M = V^{-1}$ is the matrix with entries

$$M(p, q) = (-1)^{n-q} \sigma_{n-q}^p(x) Q_{j \neq q}(x_q - x_j)$$
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$$V := \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{bmatrix}$$

with no 0 entries. Then its inverse $M := V^{-1}$ is the matrix with entries

$$M(p, q) = \frac{(-1)^{n-q} \sigma_{n-q}^p(x)}{\prod_{j \neq q} (x_q - x_j)}.$$
Let $z_k = e^{2\pi i \theta_k}$ be the values determining the random Vandermonde matrix. Let $P(z)$ be the polynomial defined as

$$P(z) := \prod_{k=1}^{n} (z - z_k)$$
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- Using same basic matrix norm inequalities
- Together with the fact that the coefficients $\sigma^p_r$ are the coefficients (up to sign) of the polynomial $P(z)/(z - z_p)$

We can show that:

$$\lambda_1 \leq \frac{n^2}{\max_{|z|=1} |P(z)|^2}$$
Theorem (T., Whiting)

Given $\epsilon > 0$ we have that

$$\mathbb{P} \left( \max_{|z|=1} |P(z)|^2 \geq \exp(\sqrt{\pi\epsilon}\sqrt{n}/2) \right) \geq 1 - \epsilon$$

for $n$ sufficiently large and $\gamma = \log \left( \frac{\cos(\pi/8)}{\sin(\pi/8)} \right)$. 
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Given \( \epsilon > 0 \) we have that

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Consider an increasing sequence of integers \( \{k_p\}_{p=1}^{\infty} \), Let \( V \) be the \( n \times n \) matrix

\[
V(i,j) := \frac{1}{\sqrt{n}} z_j^{k_i}
\]

where \( z_j := e^{2\pi i \theta_j} \). If \( k_p = p - 1 \) the matrix \( V \) is the usual random Vandermonde matrix.
Generalized Random Vandermonde Matrices

Consider an increasing sequence of integers \( \{k_p\}_{p=1}^\infty \), let \( \mathbf{V} \) be the \( n \times n \) matrix

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V(i, j) := \frac{1}{\sqrt{n}} z_j^{k_i}
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where \( z_j := e^{2\pi i \theta_j} \). If \( k_p = p - 1 \) the matrix \( \mathbf{V} \) is the usual random Vandermonde matrix.

**Theorem (T., Whiting)**

Let \( \{k_p\}_{p=1}^\infty \) be an increasing sequence of positive integers. Then

\[
m_r = \sum_{\rho \in \mathcal{P}(r)} K_\rho
\]

where \( \mathcal{P}(r) \) is the set of partitions of \( \{1, 2, \ldots, r\} \) and \( K_\rho := \lim_{n \to \infty} \frac{|S_{\rho, n}|}{n^{r+1-|\rho|}} \)

\[
S_{\rho, n} := \left\{ (p_1, \ldots, p_r) \in \{1, 2, \ldots, n\}^r : \sum_{i \in B_j} k_{p_i} = \sum_{i \in B_j} k_{p_i+1} \right\}
\]

There exists a unique probability measure \( \mu \) supported in \([0, \infty)\) with these moments.
Proposition

Let \( \rho \in \mathcal{P}(r) \) then \( K_\rho = 1 \) if and only if the partition \( \rho \) is non-crossing.
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Example

Let \( r = 4 \) and let \( \rho = \{\{1, 3\}, \{2, 4\}\} \). Then

\[
K_\rho = \lim_{n \to \infty} \frac{|S_{\rho, N}|}{N^3}
\]

where

\[
S_{\rho, n} = \{(p_1, p_2, p_3, p_4) \in \{1, 2, \ldots, n\}^4 : k_{p_1} + k_{p_3} = k_{p_2} + k_{p_4}\}.
\]

For the case \( k_\rho = p - 1 \) we saw that \( K_\rho = 2/3 \). For the case \( k_\rho = 2^p \) we see that

\[
S_{\rho, n} = \{(p_1, p_2, p_3, p_4) \in \{1, 2, \ldots, n\}^4 : 2^{p_1} + 2^{p_3} = 2^{p_2} + 2^{p_4}\}.
\]

For positive integers \( \{a, b, c, d\} \) the equation \( 2^a + 2^b = 2^c + 2^d \) holds if and only if \( \{a, b\} = \{c, d\} \). Therefore, \(|S_{\rho, N}| = 2n^2 - n \) and hence \( K_\rho = 0 \).
Theorem

Let $k_p = 2^p$ then for every $r$ and $\rho \in \mathcal{P}(r)$ the coefficient $K_\rho = 0$ if the partition is crossing. Hence

$$m_r = |NC(r)|$$

the number of non-crossing partitions and $\mu$ is the Marchenko–Pastur distribution

$$d\mu(x) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{[0,4]}.$$
**Figure:** The blue graph is the histogram of eigenvalues of the matrix $VV^*$ for the sequence $k_p = 2^p$ and $N = 100$ over 1000 trials. The red curve is the Marchenko–Pastur distribution.

**Figure:** Histogram of the eigenvalues of matrix $VV^*$ for the sequence $k_p = p - 1$ and $N = 100$ over 1000 trials.
Some mixed moments suggest that it might be possible that $V_n$ is $U_nH_n$ where $U_n$ is asympt. free from $H_n$!
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Also, below we see the EED of $V_n$ vs. the eed of $\Phi_n V_n$ where $\Phi_n$ is a $n \times n$ Haar distributed unitary independent from $V_n$.

**Figure:** Eigenvalues of matrix $V$ and of $\Phi V$. 
R–diagonal?

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Also, below we see the EED of $V_n$ vs. the eed of $\Phi_n V_n$ where $\Phi_n$ is a $nxn$ Haar distributed unitary independent from $V_n$.

![Eigenvalues of matrix $V$ and of $\Phi V$.](image)

**Figure:** Eigenvalues of matrix $V$ and of $\Phi V$.

Thanks!