On the Stable Paths Problem

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Abstract

The Border Gateway Protocol (BGP) is the interdomain routing protocol used to exchange routing information between Autonomous Systems (ASes) in the internet today. While intradomain routing protocols such as RIP are basically distributed algorithms for solving shortest path problems, the graph theoretic problem that BGP is trying to solve is the stable paths problem (SPP). Unfortunately, unlike shortest path problems, it has been shown that instances of SPP can fail to have a solution and so BGP can fail to converge.

We define a fractional version of SPP and show that all instances of fractional SPP have solutions. We also show that there are polynomial time reductions from a number of well known graph problems to SPP. For example, finding stable matchings in hypergraphic preference systems (a generalization of graph stable matchings to the case of hypergraphs), and computing kernels in directed graphs are both polynomial time reducible to SPP. These reductions remain valid in the fractional case. Thus the existence of a polynomial time algorithm for computing fractional solutions to SPP would imply polynomial time algorithms for fractional solutions to these other problems as well.

1 Introduction

The internet consists of tens of thousands of subdomains known as Autonomous Systems (ASes) where each AS is a network of routers controlled by some administrative agent. The managers of an AS have the conflicting desires to have their AS connected to the rest of the internet (i.e., to have routes to destinations (IP addresses) in other ASes and to have other ASes know how to route traffic to the destinations that it owns) but not to allow too much traffic of other ASes to transit over their network. In order to control these issues, neighboring ASes establish contracts called service...

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level agreements (SLAs) between each other. These contracts can roughly be thought of as promises to transit certain traffic for each other. Thus network operators need some way to encode in their routers, the routing policies that would let them meet the requirements of their SLAs. To do this, they use a protocol known as the Border Gateway Protocol (BGP) [10].

BGP can be thought of as working in the following manner. Consider some destination \( d \) where \( d \) is an IP address or more generally, a block of IP addresses. The router where \( d \) originates will announce via BGP to some of its neighbors that it “owns” \( d \). Which neighbors it tells is a function of the economic deals it has with its neighbors. In turn, these neighbors might tell some of their neighbors that a route to \( d \) is available through them. In general, a router \( R \) might hear via BGP from a subset of its neighbors, about a number of different routes to a particular destination \( d \). BGP then uses the encoded economic policies of the AS in which it belongs, to choose its most “preferred” route from amongst those it currently knows about. Then it selectively tells some neighbors about this most preferred route via BGP.

Unfortunately, the individual economic goals of an AS might result in route preferences that conflict with the preferences of other ASes in such a way that BGP never converges [13]. In order to study this phenomenon, BGP has been modeled by Griffin, Shepherd and Wilfong [5] as a formal graph theoretic problem called the Stable Paths Problem (SPP). They showed that some instances of SPP fail to have a solution, and therefore BGP can fail to converge (since in these cases there is no stable solution to which it can converge). We will describe SPP in Section 2.

Motivated by some similarities between SPP and the stable matching problem [3] and the fact that while stable matchings do not always exist, fractional stable matchings do always exist [12, 1], we define a fractional version of SPP we call fractional SPP (see Section 3). Intuitively, SPP and fractional SPP can be thought of as follows. In both, each node has a preference ordering of (some of the) paths from itself to the specified destination. In SPP, each node \( v \) “chooses” a path \( P_v \), and this allows other nodes to choose paths that contain \( P_v \) as a subpath. One can think of each \( v \) as putting a weight of 1 on \( P_v \) and 0 on all other paths from \( v \) to the destination. A solution to an SPP instance then is a set of choices in which each node selects a path that obeys the above subpath constraint and no node can change its mind (while the others remain fixed) and get a more preferred path. On the other hand, in fractional SPP a node can fractionally assign weights to paths from itself to the destination so that it assigns a total weight of no more than 1. When a node \( u \) puts a weight \( w \) on a path \( P \), this constrains the weights the other nodes can put on paths containing \( P \) so that for each other node these weights total no more \( w \). The intuition is that normally we view a node’s BGP announcement of a route \( P \) to \( d \) as saying that if another node hears of \( P \) then it is permitted to send all of its traffic to \( d \) so that it passes over \( P \). In our fractional model of BGP, we then think of allowing a node to announce routes each with a fractional weight \( w \), \( 0 \leq w \leq 1 \) where the total of the weights on routes offered by a node is at most 1. The weight can be interpreted as meaning
that the node is offering to allow any node hearing of this route to send (at most) a $w$ fraction of its traffic to $d$ along this route. The notion of stability in our fractional model is similar to SPP. Intuitively, a solution is one in which no node can shift some weight to more preferred paths given that the other nodes keep their weights fixed. In Sections 4 and 5 we show that, unlike (integral) SPP, every instance of fractional SPP has a solution. Our main tool is Scarf’s Lemma from [11]. The idea of applying Scarf’s Lemma to a stability type problem was used by Aharoni and Fleiner in [1].

One can view SPP as a pure strategy game whose solutions correspond to Nash equilibria. However it should be noted (see Section 6) that it is not the case that fractional SPP is just a mixed strategy game and so we cannot conclude that stable solutions (i.e., Nash equilibria) exist simply by appealing to Nash’s Theorem [9]. It may also be possible to formulate SPP in terms of a cooperative game in order to study its core, but it is not necessary to do so for our purposes and we therefore choose to use Scarf’s Lemma directly rather than involving the machinery of game theory.

In Section 7 we show that some familiar graph theoretic problems are polynomial time reducible to SPP. For example, we show that the hypergraph stable matching problem is reducible to SPP. Then the fact that all instances of this problem have a fractional solution, first proved by Aharoni and Fleiner in [1], follows from the result in this paper that all SPP instances have fractional solutions. We then consider the problem of finding a kernel in a directed graph. A fractional version of the kernel problem has been defined and a sufficient condition for instances to have a fractional kernel has been given by Aharoni and Holzman [2]. By reducing the kernel problem to SPP, we can conclude a similar result with a different proof. Thus fractional SPP is complete for these problems in the sense that any polynomial time algorithm for finding a solution to fractional SPP will give a polynomial time algorithm for these other problems. We remark that recently, an approximation algorithm for SPP was announced in [8].

We end the paper with a section on the complexity of computing fractional solutions to SPP. In particular we show that it is NP-hard to compute a fractional solution to SPP that maximizes social welfare, a parameter that measures “optimality” of solutions.

## 2 Formal Definition of SPP

The dynamic operation of BGP as outlined in Section 1, can be modeled as an equivalent static graph problem called the stable paths problem (SPP) defined below. The problems are equivalent in the sense that for any network and for any configuration of BGP (i.e., encoding of policies) on the routers in the network, BGP has a stable solution that it might converge to if and only if the corresponding instance of SPP has a solution.
We now describe SPP as it was defined in [5]. Let $G$ be a connected graph with a distinguished node $d$, called the destination. Suppose that for each node $v \neq d$ in $G$ there is an associated nonempty set $\pi(v)$ of simple paths from $v$ to $d$. Here $\pi(v)$ need not contain all paths from $v$ to $d$, but we do require the condition that if $P \in \pi(v)$ has a proper final segment $S$ from $y$ to $d$ then $S \in \pi(y)$. Each $v$ has a preference list of the paths in $\pi(v)$, which is just a linear order of the paths in $\pi(v)$. We write $P < P'$ to mean that $P$ and $P'$ are both in some $\pi(v)$ and $v$ prefers $P'$ to $P$. The notation $P \leq P'$ simply means $P < P'$ or $P = P'$. We describe the preferences numerically as follows: for $P \in \pi(v)$, we define $u(P)$, the utility of path $P$, to be $c$ if there are $c - 1$ paths $P' \in \pi(v)$ where $P' < P$. We define $U = \cup_v \pi(v)$. For a path $S$ we also define $\pi(v, S)$ to be the set of paths in $\pi(v)$ that end with the path $S$. (Note that if $S$ is just the trivial path $d$ then $\pi(v, S) = \pi(v)$.) A solution to an SPP instance is a (not necessarily spanning) arborescence $T$ in $G$ with sink node $d$, with the following stability property. Let $Q$ be any path, and let $v$ be its starting node. Then one of the following holds:

(a) $P \subseteq T$ for some $P \in \pi(v)$ with $P \geq Q$,

(b) there exists a proper final segment $S$ of $Q$ such that $S \not\subseteq T$.

![Figure 1: SPP instance with no solution.](image)

It is known that not every instance of SPP has a solution. For example, the instance called BAD GADGET is an instance of SPP that has no solution [6] and is described as follows (see Figure 1): $G$ is a copy of $K_4$ with nodes $a$, $b$, $c$, and $d$. Each of $a$, $b$, and $c$ has two paths in its preference list: $a$ has $P_1 = ad$ and $P_2 = abd$, and $a$ prefers $P_2$ to $P_1$. The preference lists for $b$ and $c$ are analogous: each prefers to go through its clockwise neighbor than to go straight to $d$. The preference lists are shown in the figure ordered from most preferred to least preferred in order from top to bottom. It is easily seen that no solution exists for such an instance of SPP.

### 3 Fractional SPP

In this section we define a fractional generalization of SPP that we call fractional SPP. The parameters defining an instance of fractional SPP are the same as those for an instance $(G, d, \pi(), <)$ of
SPP. That is, we have a graph $G$ with a designated destination node $d$ where each non-destination node $v$ has a preference list $\pi(v)$, an ordered list of some of the paths from itself to the destination node $d$. The only difference will be the definition of a solution which we now describe.

For fractional SPP we define a solution to be an assignment of a non-negative weight $w(P)$ to each path $P$ in $\pi(v)$ for every $v \neq d$ so that the weights satisfy the three properties listed below.

(U) “Unity” condition: For each node $v$, $\sum_{P \in \pi(v)} w(P) \leq 1$.

(T) “Tree” condition: For each node $v$, and each path $S$, we have $\sum_{P \in \pi(v,S)} w(P) \leq w(S)$.

(S) “Stability” condition: Let $Q$ be a path, and let $v$ be its starting node. Then there exists a proper final segment $S$ of $Q$ such that $\sum_{P \in \pi(v,S)} w(P) = w(S)$, and moreover each $P \in \pi(v,S)$ with $w(P) > 0$ is such that $P \geq Q$. Here $S$ could be the trivial path $d$, where for convenience we set $w(d) = 1$.

It is not difficult to show that the set of solutions to SPP precisely corresponds to the set of solutions $w$ to fractional SPP in which $w(P) \in \{0,1\}$ for every $P$ (see Section 6). Our aim is to show that every instance of fractional SPP has a solution. Because of a technicality, we will do this in two stages. First we show that for any positive constant $\epsilon$, every instance of fractional SPP has an $\epsilon$-solution, which is defined as follows. For each node $v$ and each path $P \in \pi(v)$, we assign a non-negative weight $w(P)$ such that the following conditions hold.

(U) “Unity” condition: $\sum_{P \in \pi(v)} w(P) \leq 1$ for each $v$.

(\epsilon T) “$\epsilon$-Tree” condition: For each node $v$, and each path $S$, we have $\sum_{P \in \pi(v,S)} w(P) \leq w(S) + \epsilon$.

(\epsilon S) “$\epsilon$-Stability” condition: Let $Q$ be a path, and let $v$ be its starting node. Then one of the following holds:

- $\sum_{P \in \pi(v)} w(P) = 1$, and each $P \in \pi(v)$ with $w(P) > 0$ is such that $P \geq Q$.
- there exists a proper final segment $S$ of $Q$ such that $\sum_{P \in \pi(v,S)} w(P) = w(S) + \epsilon$, and moreover each $P \in \pi(v,S)$ with $w(P) > 0$ is such that $P \geq Q$.

We remark that in the $\epsilon$-Stability condition, the case in which $S$ is the trivial path is stated separately and is stronger (it does not depend on $\epsilon$). This is done largely for convenience, and alternate definitions of an $\epsilon$-solution would also have been possible.

In the next section we show that for every $\epsilon > 0$, every instance of fractional SPP has an $\epsilon$-solution. Then in the following section we apply a standard convergence argument to conclude that every instance has an exact solution.
Approximate Solvability of Fractional SPP

The main tool in our proof is the following important result due to Scarf [11].

**Theorem 4.1. (Scarf’s Lemma)** Let \( n < m \) be positive integers, let \( b \in \mathbb{R}_+^n \), and let \( B \) and \( C \) be \( n \times m \) matrices with the following properties:

1. **(SL1)** the first \( n \) columns of \( B \) form an identity matrix,
2. **(SL2)** the set \( \{ x \in \mathbb{R}_+^m : Bx = b \} \) is bounded,
3. **(SL3)** each entry \( c_{ik} \) for \( k > n \) satisfies \( c_{ii} < c_{ik} < c_{ij} \) for each \( j \neq i, j \leq n \).

Then there exists \( x \in \mathbb{R}_+^m \) such that \( Bx = b \) and the set of columns of \( C \) that correspond to the support \( \text{supp}(x) = \{ k : x_k \neq 0 \} \) of \( x \) form a dominating set. This means that for every column \( j \), there exists a row \( i \) such that \( c_{ik} \geq c_{ij} \) for every \( k \in \text{supp}(x) \).

We will apply Scarf’s Lemma to matrices \( B \) and \( C \) defined from an instance of fractional SPP as follows.

**Definition 4.2.** Let \( (G, d, \pi(), <) \) be a given instance of fractional SPP. Let \( N = \{ (v, P) : P \subset Q \text{ for some } Q \in \pi(v) \} \) (note the proper containment), and set \( n = |N| \). We let \( m = n + t \) where \( t \) is the number of paths in \( U \). The \( n \times t \) matrix \( B' \) is indexed by \( N \cup U \), with entries as follows: the \( ((v, P), Q) \)-entry is \(-1\) if \( P = Q \), is \( 1 \) if \( Q \in \pi(v, P) \), and is \( 0 \) otherwise.

The matrix \( B \) is formed by attaching an \( n \times n \) identity matrix to \( B' \) on the left.

Let \( M \) be a number larger than the size of any preference list. The matrix \( C' \) is defined as follows: if \( Q \in \pi(v, P) \) then the \( ((v, P), Q) \)-entry is the utility \( c \) of \( Q \in \pi(v, P) \), and if \( Q \notin \pi(v, P) \) then the entry is \( M \).

The matrix \( C \) is formed by attaching a \( n \times n \) matrix to \( C' \) on the left, in which each diagonal entry is smaller than each entry of \( C' \), and each off-diagonal entry is larger than each entry of \( C' \).

Let us emphasize that the trivial path is used in the index set \( N \), even though it is not an element of the union \( U = \bigcup_v \pi(v) \). Note then in particular that when \( P \) is the trivial path, there is no \(-1\) entry in the row \( (v, P) \). Observe that the matrices \( B \) and \( C \) as defined here satisfy conditions (SL1) and (SL3) in the assumptions of Scarf’s Lemma. Given \( \epsilon \geq 0 \), we let \( b(\epsilon) \in \mathbb{R}^n \) be the vector with coordinates indexed by \( N \) defined as follows: for the trivial path \( P = d \), each \( (v, P) \)-coordinate of \( b(\epsilon) \) is \( 1 \), and for all other paths the \( (v, P) \)-coordinate is \( \epsilon \). The next lemma verifies condition (SL2) for \( B \) together with the vector \( b(\epsilon) \).
Lemma 4.3. Let \((G, d, \pi(), <)\) be an instance of fractional SPP, and let \(B, C, N, n\) and \(m\) be as defined in (4.2). Let \(\epsilon \geq 0\) be given. Then each coordinate of each element \(x\) of the set \(\{x \in \mathbb{R}^n_+ : Bx = b(\epsilon)\}\) lies in the interval \([0, 1 + \epsilon]\).

Proof. Suppose the vector \((g(v_1, d), \ldots, g(v_s, d), g(v_1, P_1), \ldots, g(v_s, P_t), w(P_1) \ldots, w(P_r))\) is a non-negative solution to \(Bx = b(\epsilon)\). (Here \(\{v_1, \ldots, v_s\}\) is the node set of \(G\), and \(U = \{P_1, \ldots, P_r\}\).) By definition of \(B\), we have first of all (looking at rows \((v, P)\) for the trivial path \(P = d\)) that for each \(v\), \(g(v, d) + \sum_{Q \in \pi(v)} w(Q) = 1\). This tells us that \(g(v, d) \leq 1\) and \(w(Q) \leq 1\) for each \(v\) and \(Q \in U\). Now for each \(P\) of length at least 1 we have \(g(v, P) - w(P) + \sum_{Q \in \pi(v, P)} w(Q) = \epsilon\), telling us that each \(g(v, P) \leq w(P) + \epsilon \leq 1 + \epsilon\). Thus each coordinate of the solution lies in the interval \([0, 1 + \epsilon]\). \(\square\)

Our last preliminary lemma is a technical result that essentially tells us that the solution provided by Scarf’s Lemma gives a stable solution to SPP.

Lemma 4.4. Let \(X\) be a dominating set of columns in \(C\). Suppose \(X\) is also the support of some non-negative solution \(x^*(\epsilon)\) to \(Bx = b(\epsilon)\) for some \(\epsilon > 0\).

Let \(x^*(\alpha) = (g_0(v_1, d), \ldots, g_0(v_s, d), g_0(v_1, P_1), \ldots, g_0(v_s, P_t), w_0(P_1) \ldots, w_0(P_r))\) be a non-negative solution to \(Bx = b(\alpha)\) for some \(\alpha \geq 0\), whose support is contained in \(X\). Then the weight function \(w_\alpha\) satisfies the \(\alpha\)-stability condition \((\alpha S)\).

Proof. Let \(Q\) be a path. Let \(x^*(\epsilon) = (g_\epsilon(v_1, d), \ldots, g_\epsilon(v_s, d), g_\epsilon(v_1, P_1), \ldots, g_\epsilon(v_s, P_t), w_\epsilon(P_1) \ldots, w_\epsilon(P_r))\). Suppose that the column in \(C'\) indexed by \(Q\) is dominated in \(X\) in the row \((v, P)\). Then \(g_\epsilon(v, P) = 0\), as the \([(v, P), (v, P)]\)-entry of \(C\) is smaller than all entries in \(C'\), and hence also \(g_\alpha(v, P) = 0\). Therefore if \(P = d\) we get \(\sum_{P' \in \pi(v, P)} w_\epsilon(P') = 1\) and \(\sum_{P' \in \pi(v, P)} w_\alpha(P') = 1\), and if \(P \neq d\) then \(\sum_{P' \in \pi(v, P)} w_\epsilon(P') = w_\epsilon(P) + \epsilon\) and \(\sum_{P' \in \pi(v, P)} w_\alpha(P') = w_\alpha(P) + \alpha\). Now we claim that \(Q \in \pi(v, P)\).

Suppose not: then the \([(v, P), Q]\) entry of \(C\) is \(M\), and so by definition of \(C'\), none of the paths \(P'\) with \(w_\epsilon(P') \neq 0\) are in \(\pi(v, P)\). But in this case \(\sum_{P' \in \pi(v, P)} w_\epsilon(P') = 0\), contradicting the fact that this value is 1 or \(w_\epsilon(P) + \epsilon > 0\). (We remark that this is where \(\epsilon > 0\) was needed.) Therefore \(Q \in \pi(v, P)\). Finally, note that by definition of \(C'\), all \(P' \in \pi(v, P)\) for which \(w_\epsilon(P') \neq 0\) are preferred by \(v\) to \(Q\) (or are equal to \(Q\)). Hence also all \(P' \in \pi(v, P)\) for which \(w_\alpha(P') \neq 0\) are preferred by \(v\) to \(Q\) or are equal to \(Q\). Thus \(w_\alpha\) satisfies \((\alpha S)\). \(\square\)

Now we are ready to use Scarf’s Lemma to show that every instance of fractional SPP has an \(\epsilon\)-solution for any \(\epsilon > 0\). As mentioned previously, the matrix \(C\) will capture the notion of stability, while the matrix \(B\) will guarantee the unity and tree conditions.

Theorem 4.5. Let \(\epsilon > 0\), and \((G, d, \pi(), <)\) be an instance \(I\) of fractional SPP. Let matrices \(B\) and \(C\) be as defined in (4.2). Then there exists a non-negative solution \(x^*\) to \(Bx = b(\epsilon)\), whose support \(S\) is dominating in \(C\). This gives an \(\epsilon\)-solution of \(I\).
Proof. By Lemma 4.3, the set \( \{ x \in \mathbb{R}^m_+ : Bx = b(\epsilon) \} \) is bounded. We may therefore apply Scarf’s Lemma to obtain a solution \( x^* = (g(v_1, d), \ldots, g(v_s, d), g(v_1, P_1), \ldots, g(v_s, P_1), w(P_1) \ldots, w(P_r)) \) whose support is dominating in \( C \). We claim that the weight function \( w \) is an \( \epsilon \)-solution to \( I \).

The unity condition \((U)\) follows because for each \( v \), we have \( g(v, d) + \sum_{Q \in \pi(v)} w(Q) = 1 \). The \( \epsilon \)-tree condition \((\epsilon T)\) holds because for each \( P \) of length at least 1 we have \( g(P) - w(P) + \sum_{Q \in \pi(v, P)} w(Q) = \epsilon \). To verify the \( \epsilon \)-stability condition \((\epsilon S)\) we apply Lemma 4.4 with \( \alpha = \epsilon \) and \( x^*(\epsilon) = x^*(\alpha) = x^* \). Therefore \( w \) is an \( \epsilon \)-solution to \( I \) as required.

\[ \square \]

5 Exact Solution

In this brief section we show that Theorem 4.5 in fact implies that every instance of SPP has a solution (i.e. a 0-solution). To find it, we will just consider an infinite sequence of \( \epsilon \)-solutions where \( \epsilon \) tends to 0, and show that some subsequence of these solutions converges to an exact solution.

Theorem 5.1. Every instance \( I \) of fractional SPP has a solution.

Proof. Let \( (G, d, \pi(), <) \) be the given instance \( I \). Let the matrices \( B \) and \( C \) be as in (4.2). For the sequence \( 1 > 2^{-1} > 2^{-2} > \ldots \) of positive constants converging to 0, consider the sequence of vectors \( b(2^{-1}), b(2^{-2}), \ldots \) as defined in Lemma 4.3.

For each \( i \geq 1 \), by Theorem 4.5 there is a non-negative solution \( x^*(2^{-i}) \) to \( Bx = b(2^{-i}) \), whose support is dominating in \( C \). Let \( S \) be a subset of columns of \( B \) that occurs as the support of \( x^*(2^{-i}) \) for infinitely many \( i \), and let \( \epsilon_1 > \epsilon_2 > \ldots \) be the infinite subsequence of \( 2^{-1} > 2^{-2} > \ldots \) for which \( S \) is the support of the solution. Since \( \epsilon_i < 1 \) for each \( i \), by Lemma 4.3, there exists a subsequence \( \alpha_1 > \alpha_2 > \ldots \) of \( \epsilon_1 > \epsilon_2 > \ldots \) such that the solutions \( x^*(\alpha_1), x^*(\alpha_2), \ldots \) converge to a vector \( x^* \), in which every coordinate lies in \([0, 2]\). The support of \( x^* \) is contained in \( S \), and by continuity \( x^* \) is a solution to \( Bx = b(0) \).

We claim that the weight function \( w \) associated with \( x^* \) is a solution to \( I \). The conditions \((U)\) and \((T)\) follow as before from the fact that \( x^* \) is a solution to \( Bx = b(0) \). To verify the stability condition \((S)\) we apply Lemma 4.4 with 0 in place of \( \alpha \) and \( \epsilon_1 \) in place of \( \epsilon \). Therefore \( w \) is a solution to \( I \) as required. \[ \square \]

6 Remarks

Consider the fractional SPP instance \( I \) whose graph and preference lists are as shown in Figure 1. If each node assigns a weight of 1/2 to each of the two paths in its preference list, it is straightforward
to verify that this is a fractional solution to \( I \) and in fact, it is the only fractional solution.

The above example might lead one to believe that perhaps there is a half-integral solution for all instances of fractional SPP just as there is for all instances of the fractional stable matching problem for graphs [12]. However this is not the case as the following example illustrates.

[Diagram of a graph showing nodes a, b, c, d, a', b', c', d' with paths labeled to illustrate the fractional solution]

Figure 2: Fractional BGP is not half-integral

Consider the fractional SPP instance shown in Figure 2. The subconfiguration consisting of \( a, b, c \) and destination node \( d \) is just the instance shown in Figure 1. That is, \( a, b \) and \( c \) all prefer the path to \( d \) through their clockwise neighbor over the direct path to \( d \). Node \( d' \) has \( \pi(d') = d'ad \).

For \( x, y \in \{a', b', c'\} \) where \( y \) is the clockwise neighbor of \( x \), \( \pi(x) \) consists of paths \( xd'ad \) and \( xyd'ad \) where \( xd'ad < xyd'ad \). Then it is straightforward to check that the only fractional solution is where the weight assigned to each of \( ad, abd, bd, bcd, cd \) and \( cad \) is \( 1/2 \) and the weight assigned to each of \( a'd'ad, a'b'd'ad, b'd'ad, b'c'd'ad, c'd'ad \) and \( c'a'd'ad \) is \( 1/4 \). Hence this instance does not have a half-integral solution.

As mentioned in the introduction, there is no mixed strategy game whose solutions in general correspond to solutions to fractional SPP. This can be seen, for instance, by the fact that the weights (probabilities) that a player in any such game would place on routes (strategies) must sum to 1 in a Nash equilibrium. However, as the previous example shows, there are instances of fractional SPP where the sum of the weights a node assigns to routes in a solution is strictly less than 1.

We conclude this section by proving the fact stated in Section 3 that solutions to SPP are in one-to-one correspondence with solutions \( w \) to fractional SPP that take only the values 0 and 1. For a solution \( T \) to SPP, we may define a solution \( w_T \) to fractional SPP by setting \( w_T(P) = 1 \) if \( P \subseteq T \) and 0 otherwise.

**Theorem 6.1.** Let an instance \( I \) of SPP be given.
1. If $T$ is a solution to SPP in $I$ then $w_T$ is a solution to fractional SPP in $I$.

2. If the solution $w$ to fractional SPP in $I$ takes only the values 0 and 1 then $T = \bigcup_{w(P)=1} P$ is a solution to SPP in $I$.

Proof. Let $T$ be a solution to SPP and set $w = w_T$. By stability of $T$ we know that if $P \subseteq T$ has starting node $v$ then $P \in \pi(v)$. Property (U) holds since $T$ is a tree. To check (T), if $v \notin T$ then $\sum_{P \in \pi(v,S)} w(P) = 0$. If $v \in T$ let $Q$ be the $(v,d)$-path in $T$. Then $w(Q) = 1$. If $S$ is a final segment of $Q$ then $S \subseteq T$ so $w(S) = 1$. Thus $Q$ is the only element of $\pi(v,S)$ with nonzero weight, and $\sum_{P \in \pi(v,S)} w(P) = 1 = w(S)$. If $S \not\subseteq Q$ then $\sum_{P \in \pi(v,S)} w(P) = 0$.

To check (S), let $Q$ be a path and let $v$ be its starting node. If (a) holds in the definition of stability of $T$ then the $(v,d)$-path $P_0$ in $T$ is the only element of $\pi(v,d)$ with nonzero weight, and $P_0 \geq Q$. Therefore $\sum_{P \in \pi(v,d)} w(P) = 1 = w(d)$ and so (S) holds for the proper final segment $d$. If (b) holds then $S \not\subseteq T$ so $w(S) = 0$ and $w(P) = 0$ for all $P \in \pi(v,S)$.

Now we prove the second statement. Suppose $w$ is a solution to fractional SPP and $w(P) \in \{0,1\}$ for every $P$. By (U), every $v$ has at most one path $P \in \pi(v)$ with $w(P) = 1$.

We claim that $T = \bigcup_{w(P)=1} P$ is a tree. It is clearly connected since every path $P$ ends at $d$. Suppose on the contrary that $T$ contains a cycle $C$. Then there exists a vertex $x$ in $C$ and two distinct paths $P_1$ and $P_2$ with $w(P_1) = w(P_2) = 1$ such that the $(x,d)$-segment $Q_1$ of $P_1$ is different from the $(x,d)$-segment $Q_2$ of $P_2$. By (T) we know $w(Q_1) \geq w(P_1) = 1$ and $w(Q_2) \geq w(P_2) = 1$. But this contradicts (U) for $x$. Thus $T$ is a tree.

To check the stability property for $T$, let $Q$ be any path and let $v$ be its starting node. Let $S$ be the proper final segment given by (S). If $S \not\subseteq T$ then (b) holds, so assume $S \subseteq T$. Suppose there exists $P \in \pi(v,S)$ with $w(P) = 1$. Then $P \subseteq T$, and by (S) $w(S) = 1$, and moreover $P \geq Q$. Thus (a) holds. We may therefore assume that $w(P) = 0$ for every $P \in \pi(v,S)$. Therefore $w(S) = 0$ by (S). But then since $S \subseteq T$ and $T$ is a tree, there must be some path $P'$ with $w(P') = 1$ such that $S$ is a final segment of $P'$. But then (T) is violated for $P'$. This contradiction completes the proof. 


7 Some Graph Problems Reducible to SPP

The instances of SPP we construct in this section will all have a particularly simple structure. We therefore begin by establishing some simple properties of such instances that can be easily referred to in our constructions. These properties will follow from our first general lemma.

Lemma 7.1. Let $(G,d,\pi(<))$ be an instance of fractional SPP and let $w$ be a solution. Let $v$ be a
node of \( G \) and suppose \( Q \in \pi(v) \). Then
\[
    w(Q) = \min_{S \subseteq Q} \{ w(S) - \sum_{P \in \pi(v,S), P > Q} w(P) \}.
\]

Proof. First we show that \( w(Q) \geq \min_{S \subseteq Q} \{ w(S) - \sum_{P \in \pi(v,S), P > Q} w(P) \} \). By the stability condition, there exists a proper final segment \( S_0 \subseteq Q \) such that \( \sum_{P \in \pi(v,S_0)} w(P) = w(S_0) \), and each \( P \in \pi(v,S_0) \) with \( w(P) > 0 \) satisfies \( P \geq Q \). Thus in particular we find
\[
    w(S_0) = \sum_{P \in \pi(v,S_0)} w(P) = \sum_{P \in \pi(v,S_0), P \geq Q} w(P).
\]
Therefore \( w(S_0) - \sum_{P \in \pi(v,S_0), P > Q} w(P) = w(Q) \), which proves our claim.

For each \( S \subseteq Q \) we know \( \sum_{P \in \pi(v,S), P > Q} w(P) + w(Q) \leq \sum_{P \in \pi(v,S)} w(P) \leq w(S) \) by (T). So \( w(Q) \leq w(S) - \sum_{P \in \pi(v,S), P > Q} w(P) \), and so we find \( w(Q) \leq \min_{S \subseteq Q} \{ w(S) - \sum_{P \in \pi(v,S), P > Q} w(P) \} \), thus completing the proof.

\[\Box\]

Many of the nodes \( v \) in our constructions will be such that the single-edge path \( vd \in \pi(v) \). We show that these nodes always satisfy the unity condition (U) with equality.

**Corollary 7.2.** Let \( (G, d, \pi(), <) \) be an instance of fractional SPP and let \( w \) be a solution. Let \( v \) be a node of \( G \) and suppose \( vd \in \pi(v) \). Then \( \sum_{P \in \pi(v)} w(P) = 1 \).

Proof. By Lemma 7.1 applied to the path \( vd \), and using the fact that \( w(d) = 1 \) for the trivial path \( d \), we find that \( w(vd) = 1 - \sum_{P \in \pi(v), P > vd} w(P) \). Hence \( \sum_{P \in \pi(v)} w(P) \geq \sum_{P \in \pi(v), P > vd} w(P) + w(vd) = 1 \), and so \( \sum_{P \in \pi(v)} w(P) = 1 \) by the unity condition.

\[\Box\]

We say that a graph \( (G, d, \pi(), <) \) forms a simple SPP instance if for every node \( v \) and every nontrivial path segment \( S \), the set \( \pi(v, S) \) has size at most one. All the graph problems we consider in this section lead to simple SPP instances, in which the following corollary is particularly useful. Here by \( Q - v \) we mean the proper final segment of \( Q \) formed by removing the initial node \( v \).

**Corollary 7.3.** Let \( (G, d, \pi(), <) \) be a simple instance of fractional SPP and let \( w \) be a solution. Let \( v \) be a node of \( G \) and suppose \( Q \subseteq \pi(v) \). Then
\[
    w(Q) = \min \{ w(Q - v), 1 - \sum_{P \in \pi(v), P > Q} w(P) \}.
\]

Proof. Let \( S_1 \subseteq Q \) be a proper final segment of \( Q \) that achieves the minimum value in \( \{ w(S) - \sum_{P \in \pi(v,S), P > Q} w(P) \} \). If \( S_1 \) is the trivial path then \( w(S_1) - \sum_{P \in \pi(v,S_1), P > Q} w(P) = 1 - \sum_{P \in \pi(v), P > Q} w(P) \). If \( S_1 \) is nontrivial then the set \( \{ P \in \pi(v, S_1) : P > Q \} \) is empty, and hence \( w(Q) = w(S_1) \). But by the tree condition (T), the minimum of \( \{ w(S_1) : S_1 \subseteq Q \} \) is attained by \( Q - v \), thus completing the proof.

\[\Box\]
7.1 Hypergraphic stable matching

In this section we show that the problem of finding fractional stable matchings in hypergraphic preference systems is a special case of finding fractional solutions to stable paths problems. In fact it is true that for any instance of the hypergraphic stable matching problem there is an equivalent instance of fractional SPP in the sense that there is a one-to-one correspondence between their solution sets. However here we give the reduction in one direction only.

A hypergraphic preference system consists of a hypergraph $G = (V, E)$ and a family of linear orders $L = \{\prec_i : u_i \in V\}$ where $\prec_i$ is an order on the hyperedges incident to $u_i$. For each $u_i \in V$, let $I(u_i)$ be the set of hyperedges in $E$ incident to $u_i$. If $h, h' \in I(u_i)$ and $h \prec_i h'$ then we say that $h'$ is preferred over $h$ by $u_i$.

The hypergraphic stable matching problem[1] is a generalization of the well known stable matching problem for graphs [3] and is defined as follows. Given a hypergraphic preference system $(G = (V, E), L)$, a solution to the hypergraphic stable matching problem is a set of hyperedges $S \subseteq E$ where

- for $h, h' \in S$, $h \cap h' = \emptyset$ (i.e. $S$ is a matching)
- for all $h \in E$ there is a $v_i \in h$ and an $m \in S$ such that $h \preceq_i m$ (i.e. $S$ is stable).

As with SPP, there are instances of the stable matching problem that have no solutions. A simple example is the triangle where each node prefers its clockwise edge over its counterclockwise edge. Thus a fractional version of this problem has been defined.

A fractional stable matching is a function $f()$ that assigns non-negative weights to the hyperedges of $E$ so that

(\textbf{F1}) “independence” condition: for every $u_i \in V$, $\sum_{h \in I(u_i)} f(h) \leq 1$, and

(\textbf{F2}) “stability” condition: for every hyperedge $h \in E$ there is a node $u_i$ of $h$ so that $f(h) + \sum_{h' \in I(u_i), h \prec_i h'} f(h') = 1$.

It was shown in [1] that every hypergraphic preference system has a fractional stable matching.

We use the following notation throughout the remainder of this section. Let $M = (G, L)$ be an instance of the hypergraphic stable matching problem where $G = (V, E)$ and $V = \{u_1, u_2, \ldots, u_n\}$. For each hyperedge $h \in E$ and each node $u_i$ of $h$, define $H_i(h) = \{h' : h \prec_i h'\}$.

Based on $M$ we will define an instance $M_{\text{SPP}} = (G', d, \pi(), <)$ of the stable paths problem. We define $G'$ to be the union over hyperedges $h \in E$ of graphs $G'(h) = (V'(h), E'(h))$ where $G'(h)$ is
defined as follows. Fix a hyperedge \( h \in E \). For ease of notation, we assume \( h = \{u_1, u_2, \ldots, u_k\} \). We let \( \{h^i_j : 1 \leq i, j \leq k\} \) be a set of \( k^2 \) new nodes, and set \( V'(h) = \{u_1, u_2, \ldots, u_k\} \cup \{d\} \cup \{h^i_j : 1 \leq i, j \leq k\} \). (Note then that nodes of the form \( h^i_j \) appear only in \( V'(h) \), while nodes \( u_i \) appear in each \( V'(h) \) with \( h \in I(u_i) \).)

Now we specify the edge set \( E'(h) \). For each \( i, 1 \leq i \leq k \) we put into \( E'(h) \) the edges of the path \( P^i(h) = h^i_1 \ldots h^i_k \). We also add the edges \( h^i_1d \) and \( u_ih^i_k \) for each \( i \). Finally, for each \( j \neq i \) we put the edge \( h^i_ju_j \) into \( E'(h) \). (We remark that the nodes \( h^i_i \) do not play an active role and are present only for notational convenience.) See Figure 3 for an illustration of the construction of \( G'(h) \) for an example hyperedge \( h = \{u_1, u_2, u_3\} \).

Next we specify the preference lists of paths for each node of \( G' \). First fix a node \( u_i \). Then \( \pi(u_i) \) consists of all paths \( u_iP^i(h)d \) with \( h \in I(u_i) \). The preferences are defined so that \( u_iP^i(h)d < u_iP^i(h')d \) if and only if \( h \prec_i h' \). Now consider a node of the form \( h^i_j \). The subpath \( h^i_jh^i_{j-1} \ldots h^i_1d \) of \( P^i(h)d \) is in \( \pi(h^i_j) \) and is the least preferred. When \( i = j \) this is the only path in \( \pi(h^i_j) \). For \( i \neq j \), the rest of \( \pi(h^i_j) \) consists of all paths \( h^i_ju_jP^j(h')d \) for \( u_j \in h, j \neq i \) and \( h \prec_j h' \) (i.e., \( h' \in H_j(h) \)). The preferences are defined so that \( h^i_ju_jP^j(h')d < h^i_ju_jP^j(h'')d \) if and only if \( h' \prec_j h'' \). Note then that \( M_{\text{SPP}} \) is a simple instance of SPP.

For a path \( w() \) of \( M_{\text{SPP}} \) and any subset of paths \( Y \subseteq \pi(u) \) from some node \( u \in V' \), let \( w(Y) = \sum_{P \in Y} w(P) \). Define \( PH_i(h) = \{u_iP^i(h')d : h' \in H_i(h)\} \). That is, a path \( u_iP^i(h')d \) is in \( PH_i(h) \) if and only if \( h \prec_i h' \).

Now we are ready to show that hypergraphic stable matching problems are, in a sense, stable paths problems.
**Theorem 7.4.** Let $M$ be an instance of hypergraphic stable matching, and let $M_{SPP}$ be the instance of the stable paths problem defined above. Then every solution $w()$ to $M_{SPP}$ gives a solution $f_w$ to $M$.

**Proof.** Suppose $w()$ is a stable solution of $M_{SPP}$. Consider Figure 4. We aim to show that for each $h$ we have $w(u_iP_j(h)d) = w(u_jP_i(h)d)$ for all $1 \leq i, j \leq k$ (that is, for all $u_i, u_j \in h$). Then we will define $f_w(h)$ to be this common value.

**Claim 1.** For all $h, i, j$ and $h' \in H_i(h)$ we have $w(h_i'uj_jP_i(h')d) = w(u_jP_i(h')d)$.

**Proof of Claim 1.** Suppose not, and let $h_i'uj_jP_i(h')d$ be the most preferred path in $\pi(h_i')$ with $w(h_i'uj_jP_i(h')d) < w(u_jP_i(h')d)$. Then by Corollary 7.3 we know $w(h_i'uj_jP_i(h')d) = 1 - \sum_{P > h_i'uj_jP_i(h')d} w(P)$. By the definition of preferences for $h_i'$ we know $\{P \in \pi(h_i') : P > h_i'uj_jP_i(h')d\} = \{h_i'Q : Q \in \pi(u_j), Q > u_jP_i(h')d\}$. By choice of $h_i'uj_jP_i(h')d$ we then know that $\sum_{P > h_i'uj_jP_i(h')d} w(P) = \sum_{Q > u_jP_i(h')d} w(Q)$. Thus we get that $w(u_jP_i(h')d) > 1 - \sum_{Q > u_jP_i(h')d} w(Q)$, implying that $\sum_{Q \in \pi(u_j)} w(Q) > 1$. This contradicts (U). Therefore $w(h_i'uj_jP_i(h')d) = w(u_jP_i(h')d)$ as claimed.

For $1 \leq i \leq k$ we let $w(PH_i(h)) = \epsilon_i$ as shown in Figure 4.

**Claim 2.** For all $h, i, j$ we have $w(h_j' \ldots h_i'd) = 1 - \max\{\epsilon_s : 1 \leq s \leq j, s \neq i\}$.

**Proof of Claim 2.** We prove this claim by induction on $j$. Let $j = 1$. If $i = 1$ then $w(h_i'd) = 1$ by
Corollary 7.2 as required. Thus we may assume \( i \neq 1 \). Since \( h_i^1 d \) is the least preferred path in \( \pi(h_i^1) \), by Claim 1 and Corollary 7.2 we have 

\[
 w(h_i^1 d) = 1 - \sum_{P > h_i^1 d} w(P) = 1 - \sum_{h' \in H_i(h)} w(h_i^1 u_1 P^1(h') d) = 1 - \sum_{h' \in H_i(h)} w(u_1 P^1(h') d) = 1 - \epsilon_1. 
\]

This establishes the base case \( j = 1 \). Now for \( j > 1 \), if \( j = i \) then \( h_i^j \ldots h_i^1 d \) is the only path in \( \pi(h_i^j) \), so \( w(h_i^j \ldots h_i^1 d) = w(h_i^j \ldots h_i^1 d) \) by Corollary 7.3, and so \( w(h_i^j \ldots h_i^1 d) = 1 - \max\{\epsilon_s : 1 \leq s \leq j, s \neq i\} \) by the induction hypothesis. Thus we assume \( j \neq i \). Then by Corollary 7.3 we have 

\[
 w(h_i^j \ldots h_i^1 d) = \min\{w(h_i^j \ldots h_i^1 d), 1 - \sum_{P > h_j^j \ldots h_i^1 d} w(P)\}. 
\]

But \( \{P > h_j^j \ldots h_i^1 d\} = \{h_j^j u_j P_j(h') d : h' \in H_i(h)\} \) by the definition of preferences, so by Claim 1 

\[
 \sum_{P > h_j^j \ldots h_i^1 d} w(P) = \sum_{h' \in H_i(h)} w(u_j P_j(h') d) = \epsilon_j. 
\]

Thus using the induction hypothesis \( w(h_j^j \ldots h_i^1 d) = \min\{w(h_{j-1}^j \ldots h_i^1 d), 1 - \epsilon_j\} = \min\{1 - \max\{\epsilon_s : 1 \leq s \leq j - 1, s \neq i\}, 1 - \epsilon_j\} \). Therefore 

\[
 w(h_j^j \ldots h_i^1 d) = 1 - \max\{\epsilon_s : 1 \leq s \leq j, s \neq i\} \text{ as required.} 
\]

Since \( P^i(h) = h_k^i \ldots h_i^1 d \) we find in particular that \( w(P^i(h)) = 1 - \max\{\epsilon_s : 1 \leq s \leq k, s \neq i\} \).

**Claim 3.** For all \( h, i \) we have 

\[
 w(u_i P^i(h) d) = 1 - \max\{\epsilon_s : 1 \leq s \leq k\}. 
\]

**Proof of Claim 3.** By Corollary 7.3 and the definition of preferences 

\[
 w(u_i P^i(h)) = \min\{w(P^i(h)), 1 - \sum_{h' \in H_i(h)} w(u_i P^i(h') d)\}. 
\]

Therefore by Claim 2 and the definition of \( \epsilon_i \) we conclude 

\[
 w(u_i P^i(h)) = 1 - \max\{\epsilon_s : 1 \leq s \leq k\}.
\]

Claim 3 then tells us that for each \( h \), every \( w(u_i P^i(h)) \) for \( u_i \in h \) takes the same value. We then define a function \( f_w() \) on \( E \) by setting \( f_w(h) \) to be this value \( 1 - \max\{\epsilon_s : 1 \leq s \leq k\} \}. \) We claim that \( f_w() \) is a fractional stable matching for \( M \). The independence condition (F1) is immediate from the definition since each \( \epsilon_s \) is non-negative. To verify condition (F2), let \( h = \{u_1, \ldots, u_k\} \) be an arbitrary edge. Suppose without loss of generality that \( \epsilon_1 = \max\{\epsilon_s : 1 \leq s \leq k\} \). Then \( f_w(h) = 1 - \epsilon_1 \). But by definition of \( f_w() \) we see \( \epsilon_1 = \sum_{h' \in H_i(h)} w(u_i P^i(h') d) = \sum_{h' \in H_i(h), h < i h'} f_w(h') \). Therefore the node \( u_i \in h \) satisfies (F2). Thus \( f_w() \) is a fractional stable matching for \( M \).

Notice that as a consequence of Theorem 7.4, the result of [1] stating that all instances of hypergraph stable matching have a fractional solution follows as a corollary of Theorem 5.1. Moreover, note that the construction \( M_{\text{SPP}} \) is polynomial in the size of \( M \). Thus a polynomial time algorithm for computing fractional solutions to instances of SPP would imply a polynomial time algorithm to compute fractional stable matchings for hypergraphic preference systems.

### 7.2 Kernels

Let \( D = (V, A) \) be a directed graph (digraph) such that each pair of nodes has at most one arc joining them in each direction. A kernel \( S \) of \( D \) is an independent subset of \( V \) (meaning for all \( u, v \in S, (u, v) \not\in A \) and \( (v, u) \not\in A \) that is absorbing (meaning if \( v \in V \setminus S \) then there exists a node \( s \in S \) such that arc \((v, s) \in A \). In what follows, for \( u \in V \) we use the notation \( H(u) = \{v : (u, v) \in A\} \).
and $H^+(u) = \{u\} \cup H(u)$. An arc $(u, v) \in A$ is said to be an irreversi-
ble arc if $(v, u) \not\in A$.

Not every digraph has a kernel, for example the directed cycle of length three. Thus fractional
kernels have been considered. A fractional kernel is a function $f : V \to [0, 1]$ such that

- for every clique $K$ in $D$, $\sum_{v \in K} f(v) \leq 1$, and
- for all $v \in V$, $\sum_{u \in H^+(v)} f(u) \geq 1$.

Again the directed cycle of length three shows that not all digraphs have fractional kernels, but
Aharoni and Holzman [2] proved the following.

**Theorem 7.5.** Let $D$ be a digraph in which no clique $K$ contains a cycle of irreversi-
able arcs. Then $D$ has a fractional kernel.

In fact they showed that a stronger form of fractional kernel (the so-called strong fractional kernel,
see [2]) always exists in such digraphs. Here we consider a somewhat different version of Theorem 7.5.
Let us say that a subset $W$ of $V$ is co-acyclic if it induces in $D$ a clique with no cycle of irreversi-
able arcs. (It is easy to check that this is equivalent to $W$ inducing an acyclic digraph in the comple-
ment $\bar{D}$ of $D$, hence the term co-acyclic.) We define a fractional co-acyclic kernel in $D$ to be a function
$f : V \to [0, 1]$ such that

(C1) for every co-acyclic set $W$ in $D$, $\sum_{v \in W} f(v) \leq 1$, and

(C2) for all $v \in V$, $\sum_{u \in H^+(v)} f(u) \geq 1$.

Here we show that the problem of finding a fractional co-acyclic kernel can be interpreted as
an instance of fractional SPP, and hence prove by Theorem 5.1 that every digraph has a fractional
co-acyclic kernel. In particular this gives an alternate proof of Theorem 7.5.

Let $D = (V, A)$ be a digraph. Then define the SPP instance $I_D = (G = (V', E), d, \pi, <)$ as
follows. The node set $V'$ is defined as $V' = V \cup \{d\}$. There is an edge $\{u, v\} \in E$ if there is an
arc $(u, v)$ or $(v, u)$ in $A$. Also, for each $v \in V$ there is an edge $\{v, d\} \in E$. The preference list
for node $v$ consists of the path $vd$ and any path $vud$ where $(v, u) \in A$. Then $v$ prefers any path of
the form $vud \in \pi(v)$ over the path $ud$ and the relative ordering amongst the length two paths in $\pi(v)$ is
arbitrary. Let $w()$ denote a weight function on the paths in $\cup_{v \in V} \pi(v)$. Then for any such $w$, define
the weight function $f_w()$ on the nodes of $D$ so that $f_w(v) = w(vd)$.

**Theorem 7.6.** If $w()$ is a fractional solution to the SPP instance $I_D$ then $f_w()$ is a fractional
co-acyclic kernel of $D$.
Proof. First we verify Condition (C2). Let $v$ be a node of $D$. Since $vd \in \pi(v)$, we find $\sum_{P \in \pi(v)} w(P) = 1$ by Corollary 7.2. By (T) and the definition of $f_w()$ we know $\sum_{P \in \pi(v)} w(P) \leq \sum_{u \in H^+(v)} f_w(u)$. Thus $\sum_{u \in H^+(v)} f_w(u) \geq 1$, satisfying (C2).

Now we consider (C1). Let $W$ be a co-acyclic set in $D$, and suppose on the contrary that $\sum_{u \in W} f_w(u) > 1$. Let $U = \{ u \in W : f_w(u) > 0 \}$, so then $\sum_{u \in U} f_w(u) > 1$. Fix $u \in U$. By Corollary 7.2 we know that $\sum_{P \in \pi(u)} w(P) = 1$, and so $\sum_{P \in \pi(u) \setminus ud} w(P) = 1 - w(ud)$. We claim that there exists some $x \in U$ such that $(u, x) \notin A$. Suppose the contrary, then $\sum_{x \in H(u)} f_w(x) \geq \sum_{x \in U \setminus \{u\}} f_w(x) > 1 - f_w(u) = 1 - w(ud)$. Therefore $\sum_{P \in \pi(u) \setminus ud} w(P) < \sum_{x \in H(u)} f_w(x)$, in other words $\sum_{P \in \pi(u) \setminus ud} w(P) < \sum_{P \in \pi(u)} w(P - u)$. Thus for some path $Q \in \pi(u) \setminus ud$ we must have $w(Q) < w(Q - u)$. Since $G$ is a simple instance of SPP, by Corollary 7.3 we see that $w(Q) = \min\{w(Q - u), 1 - \sum_{P \in \pi(u), P > Q} w(P)\} = 1 - \sum_{P \in \pi(u), P > Q} w(P)$. Thus $\sum_{P \in \pi(u), P > Q} w(P) = 1$. But $ud \in \pi(u)$ is such that $ud < Q$ and $w(ud) > 0$, which implies $\sum_{P \in \pi(u)} w(P) > 1$, contradicting the unity condition. Therefore for some $x$ we must have $(u, x) \notin A$, so since $W$ is co-acyclic $(x, u) \in A$. Since this holds for each $u \in U$, we conclude that $U$ contains a cycle of irreversible arcs, contradicting the assumption that $W$ was co-acyclic. Thus we find $\sum_{u \in W} f_w(u) \leq 1$, completing the proof. $\square$

It follows immediately from Theorems 7.6 and 5.1 that every digraph has a fractional co-acyclic kernel, thus giving another proof of Theorem 7.5. We remark that Aharoni and Holzman’s proof of (a stronger form of) Theorem 7.5 uses a different application of Scarf’s Lemma, in which the rows of the matrices $B$ and $C$ are indexed by the set of maximal cliques in $D$, which can be a very large set. Our proof applies Scarf’s Lemma using the row index set $R = \{(v, S) : S \subset P$ for some $P \in \pi(v)\}$, which has size $|V|(|A| + 1)$, and the column index set $R \cup \bigcup_{v \in V} \pi(v)$ which has size $2|V|(|A| + 1)$.

Note that the construction used to obtain an SPP instance from an instance of the kernel problem in Theorem 7.6 is polynomial in the size of the kernel instance. Thus as in the case of hypergraphic preference systems, a polynomial time algorithm for computing fractional solutions to SPP instances would imply a polynomial time algorithm for computing fractional co-acyclic kernels (and hence fractional kernels, under the assumptions of Theorem 7.5.)

8 Optimal Solutions

Consider an instance $I = (G, d, \pi(), <)$ of fractional SPP where $G = (V, E)$. Here we define a utility function to be any non-negative valued function $u()$ on paths in $I$ with the property that for any $v \in V$ and any paths $P, Q \in \pi(v)$, $u(P) < u(Q)$ if and only if $Q$ is preferred over $P$ by $v$. The payoff $p(v)$ for node $v \in V$ for a given weight assignment $w()$ is defined to be 0 if the weights of the paths in $\pi(v)$ violate the tree condition (T) and otherwise $p(v) = \sum_{P \in \pi(v)} w(P)u(P)$. Then we define the social welfare of a weighting as $\sum_{v \in V} p(v)$.
As mentioned, an \(x\) satisfying \(Bx = b(0)\) as described in Section 4 defines a weighting \(w()\) of paths that satisfies both the conditions (U) and (T). Consider the LP whose constraints are given by \(Bx = b(0)\) with the objective function \(\max_x \sum_{v \in V} p(v)\). Then solutions to this LP give path weights that satisfy (U) and (T) while maximizing the social welfare. Thus we can solve this maximum social welfare problem in polynomial time by solving such an LP. However, we show below that computing a fractional solution to an instance of SPP that maximizes the social welfare is NP-complete. That is, adding the stability constraints (S) to the unity and tree constraints changes the problem from being polynomially time solvable to being NP-hard.

![Figure 5: The graph for a variable gadget.](image)

We define an instance of fractional SPP that we show has exactly two solutions both of which are in fact integral solutions. The instance has nodes \(x, \hat{x}, \bar{x}\) and designated destination node \(d\) (see Figure 5). The preference list for node \(x\) from highest preference to lowest is \(x\hat{d}, x\bar{d}\) and \(xd\). The preference list ordered from most to least preferred for node \(s\) where \(s\) is either \(\bar{x}\) or \(\hat{x}\) is \(s\hat{d}\) followed by \(s\bar{d}\). We call such an instance a variable gadget.

**Lemma 8.1.** A variable gadget has exactly two solutions and both are integral.

**Proof.** It is straightforward to check that setting the weights so that \(w(\bar{xd}) = w(\hat{xd}) = w(x\hat{d}) = 1\) and all others are 0 satisfies unity, tree and stability conditions and hence is a solution. Similarly the weighting with \(w(xd) = w(\hat{xd}) = w(\bar{xd}) = 1\) and the rest set to 0 is another solution.

Now we just need to show that these are the only two solutions. It is simple to check that the only solutions with \(w(xd)\) equal to 0 or 1 are the two described above. So if there is another solution it must be that \(w(xd) = \epsilon\) for some \(0 < \epsilon < 1\). Then by Corollary 7.3 applied to the path \(\hat{xd}\) we find \(w(\hat{xd}) = \epsilon\). Then by Corollary 7.2 we know \(w(\hat{d}) = 1 - \epsilon\). Similary \(w(\bar{xd}) = 1 - \epsilon\). By Corollary 7.3 applied to the path \(x\hat{d}\) we see \(w(x\hat{d}) = w(\hat{d}) = 1 - \epsilon\) since \(\{P \in \pi(x) : P > x\hat{d}\} = \emptyset\). But now Corollary 7.3 applied to the path \(x\bar{d}\) tells us that \(w(x\bar{d}) = \min\{w(\bar{d}), 1 - w(x\hat{d})\} = \min\{1 - \epsilon, \epsilon\} > 0\). But then \(w(xd) + w(x\hat{d}) + w(x\bar{d}) > 1\), contradicting the unity condition. Therefore there are only two solutions for the variable gadget.

Note that Lemma 8.1 implies that the set of fractional solutions for an instance of SPP need not be convex. In particular, this says that not all instances of fractional SPP have a feasible solution
space that can be defined by a set of linear inequalities (i.e., the space of solutions need not be a convex polytope).

We will use variable gadgets in a construction showing that finding a fractional solution to an instance of SPP that maximizes the social welfare is NP-hard.

**Theorem 8.2.** Given an instance $I$ of SPP and an integer $k > 0$, the problem of computing a fractional solution to $I$ with social welfare at least $k$ is NP-hard.

**Proof.** We use a reduction from 3-sat [4]. That is, given an instance $T$ of 3-sat we construct an instance of SPP that has a fractional solution with social welfare of at least $k$ (where $k$ is defined below) if and only if $T$ has a satisfying assignment.

![Figure 6: Clause construction.](image)

For each variable $x$ in $T$ our construction will have a variable gadget as defined above. As shown in Lemma 8.1 in any solution either

(i) $w(xd) = 1$ and $w(\bar{x}d) = 0$ or

(ii) $w(xd) = 0$ and $w(\bar{x}d) = 1$.

Case (i) will indicate setting $x$ to True and case (ii) will indicate setting $x$ to False. In either case, the value of $w(\bar{x}d)$ is irrelevant in this construction (but of course will be the same as $w(xd)$).

For each clause $C$ of $T$ we choose an arbitrary but fixed ordering $C = l_1 \lor l_2 \lor l_3$ of its literals, and we add a node labeled $C$ in the construction (see Figure 6). Here each $l_i$ is either $x$ or $\bar{x}$ for some variable $x$. Then $\pi(C)$ consists of the paths $Cl_id$, $1 \leq i \leq 3$. The preference ordering on these paths is $Cl_id < Cl_jd$ if $i < j$. Notice then that in a solution, $w(Cl_id) = 0$ if $w(l_id) = 0$ by the tree
condition. If some $w(l_i d) = 1$ then let $j$ be the largest index for which $w(l_j d) = 1$, then we claim $w(Cl_j d) = 1$. This follows from Corollary 7.3 applied to $Cl_j d$, since $\{P \in \pi(C) : P > Cl_j d\} = \emptyset$. Therefore either $w(Cl_i d) = 0, 1 \leq i \leq 3$ (if $w(l_i d) = 0$ for $1 \leq i \leq 3$), or there is exactly one $i, 1 \leq i \leq 3$, such that $w(Cl_i d) = 1$.

Suppose there are $n$ variables and $m$ clauses in $T$. Define $M$ to be some integer bigger than $5n + 2(m - 1)$. To be specific, set $M = 6n + 2m$. Then we define the utilities of the paths as follows. For nodes $s \in \{\bar{x}, \hat{x}\}$ in a variable gadget define $u(sxd) = 2$ and $u(sd) = 1$. For node $x$ of a variable gadget, let $u(xd) = 1, u(xx\hat{d}) = 2$ and $u(x\hat{d}d) = 3$. For a node $C$ representing the clause $C = l_1 \lor l_2 \lor l_3$, define $u(Cl_1 d) = M, u(Cl_2 d) = M + 1$ and $u(Cl_3 d) = M + 2$.

Let $k = mM$. Notice that the total payoff for nodes in a variable gadget is at most 5 by Lemma 8.1. Therefore the contribution of all nodes in variable gadgets to the total payoff (i.e., the social welfare) is at most $5n$. The total payoff of the nodes representing clauses is at most $t(M + 2)$ if $t$ is the number of clause nodes that have some path with weight 1. Thus if $t < m$ then the total payoff will be at most $5n + (m - 1)(M + 2)$. But

$$5n + (m - 1)(M + 2) = M(m - 1) + 5n + 2(m - 1)$$
$$< M(m - 1) + M$$
$$= mM = k.$$ 

However if all clause nodes have a path with weight 1, then the total payoff will be more than $mM = k$. Therefore the total payoff is at least $k$ if and only if each clause node has a path of weight 1. But as discussed above, a clause node will have a path with weight 1 if and only if for at least one of the literals $l_i$ in the clause we have $w(l_i d) = 1$. Thus viewing setting $w(l_i d) = 1$ as equivalent to setting $l_i$ to True we get that there is a fractional solution to the constructed SPP instance with social value at least $k$ if and only if there is a satisfying assignment for the instance $T$. \hfill \Box

Note that the proof of NP-hardness might not hold for other definitions of the utility function and the consequent definitions of payoff and social welfare. However, for example it is not difficult to modify the construction above to show that NP-hardness also holds for the natural utility function mentioned in Section 2 that assigns 1 to the least preferred path for each node, 2 for the second least preferred path and so on.

References


