A Characterization of Graphs with Fractional Total Chromatic Number Equal to $\Delta + 2$

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Abstract

For a simple graph of maximum degree $\Delta$, the complexity of computing the fractional total chromatic number is unknown. Kilakos and Reed proved it lies between $\Delta + 1$ and $\Delta + 2$, and so we can approximate it within 1. In this paper, we strengthen this by characterizing exactly those simple graphs with fractional total chromatic number $\Delta + 2$. This yields a simple linear-time algorithm to determine whether a given graph has fractional chromatic number $\Delta + 2$.

1. Introduction

Colouring problems are notoriously difficult, indeed even approximating the chromatic number of an $n$-vertex graph to within a factor of $n^{1/2-\epsilon}$, $\epsilon > 0$, is NP-hard [2]. Like all NP problems, they can be formulated as integer programs and subjected to a two-pronged attack: we first solve the fractional relaxation and then use this solution to solve or obtain an approximation to the integer program. Of course, for this approach to work, we need to be able to solve or approximate the fractional relaxation. Unfortunately the fractional chromatic number is also NP-hard to approximate to within a factor of $n^{1/2-\epsilon}$ for any $\epsilon > 0$ since it is within a log $n$ factor of the chromatic number [7].

In stark contrast, we can solve the fractional relaxation of the edge colouring problem in polynomial time. For an integer $\beta$, an edge $\beta$-colouring is a covering of the edges of $G$ with $\beta$ matchings of $G$ such that every edge is in exactly one matching. Equivalently, an edge $\beta$-colouring of a graph $G = (V, E)$ is a feasible solution of value $\beta$ to the integer program

$$\min \left\{ \sum_{M \in \mathcal{M}(G)} w_M : w_M \in \{0, 1\}; \sum_{M \ni e} w_M \geq 1, e \in E, \ M \in \mathcal{M}(G) \right\},$$

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where $\mathcal{M}(G)$ is the set of all matchings of $G$. The chromatic index $\chi'(G)$ of $G$ is then the optimal value to this integer program. Even though Vizing’s Theorem guarantees $\chi'(G)$ to be one of two values, $\Delta$ or $\Delta + 1$ where $\Delta$ is the maximum degree of $G$, it is NP-complete to decide which of these two is correct.

In an attempt to strengthen Vizing’s Theorem, for a nonnegative real number $\beta$, define a fractional edge $\beta$-colouring of $G$ as a feasible solution of value $\beta$ to the linear program

$$\min \left\{ \sum_{M \in \mathcal{M}(G)} w_M : w_M \geq 0; \sum_{M \ni e} w_M \geq 1, \ e \in E, \ M \in \mathcal{M}(G) \right\}. \tag{1}$$

The fractional chromatic index $\chi'_f(G)$ of $G$ is then the optimal value to this linear program. (Precisely, the fractional relaxation requires that $\sum_{M \ni e} w_M = 1$ but the replacement of this equality with $\sum_{M \ni e} w_M \geq 1$ yields an equivalent formulation in the sense that given any feasible solution of one we can easily find a feasible solution to the other with the same value.) A simple renormalization shows that a graph can be fractionally edge $\beta$-coloured precisely when $(\frac{1}{\beta}, \frac{1}{\beta}, \ldots, \frac{1}{\beta})$ is a convex combination of incidence vectors of matchings of $G$. Using this fact, Edmond’s characterization of the matching polytope [3], and the ellipsoid method [10], one can optimally fractionally edge colour a graph in polynomial time [9]. More strongly, one can optimize any linear weight function over the matching polytope. Moreover, all of these results hold for multigraphs.

It is conjectured that the chromatic index $\chi'(G)$ of every multigraph $G$ is within one of the fractional chromatic index $\chi'_f(G)$ [4, 11], which seems plausible since it is known that $\chi'(G) = (1 + o(1))\chi'_f(G)$ [5].

In comparison to both the vertex and edge colouring problems, the computational complexity of the fractional total colouring problem lies between these two extremes. For a simple graph $G = (V, E)$, a total stable set $T \subseteq V \cup E$ is the union of a stable set $S$ of $G$ and a matching $M$ of $G$ such that no edge in $M$ has endpoints in $S$. For an integer $\beta$, a total $\beta$-colouring of a graph $G = (V, E)$ is a covering of the vertices and edges of $G$ with $\beta$ total stable sets such that every vertex and edge is in exactly one total stable set. Equivalently, a total $\beta$-colouring of a graph $G = (V, E)$ is a feasible solution of value $\beta$ to the integer program

$$\min \left\{ \sum_{T \in \mathcal{T}(G)} w_T : w_T \in \{0, 1\}; \sum_{T \ni x} w_T \geq 1, \ x \in V \cup E, \ T \in \mathcal{T}(G) \right\},$$

where $\mathcal{T}(G)$ is the set of all total stable sets of $G$. The total chromatic number $\chi''(G)$ of $G$ is then the optimal value to this integer program. Analogous to Vizing’s Theorem, the well-known Total Colouring Conjecture of Behrad [1] and Vizing [12] states that $\chi''(G)$ is one of two values: $\Delta + 1$ or $\Delta + 2$.

In contrast to the fractional chromatic index, we do not know the complexity of computing the fractional total chromatic number of a graph $G$. For a
nonnegative real number $\beta$, a fractional total $\beta$-colouring is feasible solution of value $\beta$ to the linear program

$$\min \left\{ \sum_{T \in \mathcal{T}(G)} w_T : w \geq 0; \sum_{T \ni x} w_T \geq 1, \ x \in V \cup E, \ T \in \mathcal{T}(G) \right\}. \quad (2)$$

The fractional total chromatic number $\chi''_f(G)$ of $G$ is the optimal value to this linear program. Since one weight function is equivalent to the vertex colouring problem, we cannot hope to optimize over every linear weight function. Though, we can approximate the fractional total chromatic number of any multigraph within 2, because it exceeds the fractional chromatic index by at most that amount. Kilakos and Reed strengthen this result by showing that simple graphs have fractional total chromatic number at most $\Delta + 2$ [6], thereby proving a fractional version of the Total Colouring Conjecture.

In this paper, we characterize exactly those simple graphs which have fractional total chromatic number $\Delta + 2$. This yields a simple linear-time algorithm to determine whether a given graph has fractional chromatic number $\Delta + 2$. We prove

**Theorem 1.** The fractional total chromatic number of a simple connected graph $G$ is $\Delta + 2$ precisely when $G = K_{2n}$ or $G = K_{n,n}$ for some integer $n \geq 1$.

2. Overview

The easy direction of Theorem 1 is showing that $K_{2n}$, $n \geq 1$, has fractional total chromatic number $2n + 1$ and $K_{n,n}$, $n \geq 1$, has fractional chromatic number $n + 2$. To prove the other direction of the theorem, we strengthen the result of Kilakos and Reed, proving

**Lemma 2.** For every connected simple graph which is neither $G = K_{2n}$ nor $G = K_{n,n}$ for $n > 1$ and contains at least one edge, then for each edge $e \in E$ there exists a fractional total $(\Delta + 2)$-colouring $w^e$ of $G$ such that the weight of the stable sets containing $e$ is strictly greater than 1.

Combining this result with LP duality and the easy fact that every graph has chromatic number at most $\Delta + 1$ easily yields the hard direction of Theorem 1. We omit this proof in this version. Thus, the key to Theorem 1 is Lemma 2. In the rest of this extended, we sketch the approach.

To begin, we show how to fractionally total $(\Delta + 3)$-colour any simple graph $G$. We start by arbitrarily choosing a vertex $(\Delta + 3)$-colouring of $G$. Now, Vizing’s theorem guarantees us that for each colour class $S_i$, $1 \leq i \leq \Delta + 3$, the induced subgraph $G - S_i$ has an edge $(\Delta + 1)$-colouring $y^i$. For each colouring $y^i$ and each matching $M_j \in \mathcal{M}(G - S_i)$ we let the total stable set $T_{ij} = S_i \cup M_j$ have weight $w_{T_{ij}} = 1/(\Delta + 1)$. For all other total stable sets $T \in \mathcal{T}(G)$, we let $w_T = 0$. Each $v$ in $S_i$ is in $T_{ij}$ for all $j$ between 1 and $\Delta + 1$. Each $e = xy$ is in $T_{ij(e,i)}$ for some $j(e,i)$ for every $i$ between 1 and $\Delta + 3$ such that $x \not\in S_i$ and
Thus, this is a fractional total colouring and it is immediate that the total weight is $\Delta + 3$.

If $G$ has a vertex colouring $\{S_1, S_2, \ldots, S_{\Delta+2}\}$ such that each $G - S_i$ is fractionally edge $\Delta$-colourable then the same approach yields a fractional total $(\Delta + 2)$-colouring. Kilakos and Reed obtain a vertex $(\Delta + 2)$-colouring such that $G - S_i$ is fractionally edge $\Delta$-colourable for each $i$ between 1 and $\Delta + 1$ and such that $S_{\Delta+2}$ has some special properties which allow them to fractionally total $(\Delta + 2)$-colour $G$, even though $G - S_{\Delta+2}$ may not be fractionally edge $\Delta$-colourable. Before describing their exact result and our modifications we turn to a characterization of graphs with fractional chromatic index $\Delta$ and some related observations and definitions.

We start with Edmonds’ theorem which implies that

$$\chi'_f(G) = \max \left\{ \Delta, \max_{H \subseteq G} \Lambda(H) \right\},$$

where $\Lambda(H) = |E(H)|/|V(H)|/2$. So, $G$ has fractional chromatic index $\Delta$ precisely when $G$ contains no subgraph $O$ such that $\Lambda(O) > \Delta$. This suggests the following definition. An induced subgraph $O$ of $G$ is called overfull if

$$|E(O)| > \Delta \left|V(O)\right| - 1.$$  

The work of Padberg and Rao [9] yields a polynomial time algorithm for deciding if a graph contains an overfull subgraph, and if the graph contains an overfull subgraph, finding it in polynomial time.

An overfull subgraph of $G$ is called minimal if no proper subset of its vertices induces an overfull subgraph of $G$. Kilakos and Reed showed that any two minimal overfull subgraphs of $G$ are vertex-disjoint. Let $G$ be a subgraph and $U$ be an overfull subgraph of $G$. Now, as any two minimal overfull subgraphs are disjoint, every minimal overfull subgraph graph is contained in $U$ or $G - U$. Thus, we can find all minimal overfull subgraphs by recursively checking these two parts. As at each step we partition our graph into two non-empty parts, our recursion tree will have at most $n$ nodes. Thus in polynomial time, we can partition the graph into minimal overfull subgraphs $O_1, O_2, \ldots, O_k$ and the rest $G - \bigcup_{j=1}^{k} O_j$ containing no overfull subgraphs. Henceforth, we will assume that we are given this decomposition with the additional property that $|V(O_j)| \leq |V(O_{j+1})|$ for $i$ between 1 and $k - 1$.

Now, by using this partition we can find the vertex colouring as in Kilakos and Reed. In particular, they find a vertex colouring $S_1, S_2, \ldots, S_{\Delta+2}$ such that each $O_j$ contains a vertex in colour class $S_i$ for each $i$ between 1 and $\Delta + 1$, and each $O_j$ satisfying $|V(O_j)| \geq \Delta + 2$ contains a vertex in colour class $S_{\Delta+2}$. Thus, as required, $G - S_i$ has a fractional edge $\Delta$-colouring for each $i$ between 1 and $\Delta + 1$ because $G - S_i$ contains no overfull subgraph. Moreover, this particular vertex colouring has the special properties alluded to before which allow Kilakos and Reed to complete the fractional total $(\Delta + 2)$-colouring.

To prove Lemma 2 we will add an extra condition to those imposed by Kilakos and Reed. We will ensure that $e$ is in $G - S_1$ and that $G - S_1$ has a
fractional edge $\Delta$-colouring in which the total weight of the matchings containing $e$ exceeds 1. To do so, we need to characterize graphs that have fractional edge $\Delta$-colourings such that the total weight of the matchings containing a particular edge $e$ exceeds 1. To do so, we generalize fractional edge colouring to allow for edge weights. For a nonnegative real number $\beta$, a \textit{weighted fractional edge $\beta$-colouring} of a graph $G = (V, E)$ is a feasible solution of value $\beta$ to the linear program

$$
\min \left\{ \sum_{M \in \mathcal{M}(G)} y_M : y \geq 0; \sum_{M \ni f} y_M \geq c_f, f \in E, M \in \mathcal{M}(G) \right\}, \quad (4)
$$

where $c_f$ is a nonnegative rational number (edge weight) for each $f \in E$. Notice that a fractional edge colouring is a weighted fractional edge colouring such that $c_f = 1$ for all $f \in E$. Moreover, Edmonds’ Theorem extends to the weighted case implying that the optimal value to the LP (4) is

$$
\max \left\{ \max_{u \in V} \sum_{f \in \delta(u)} c_f, \max_{H \subseteq G} \frac{\sum_{f \in E(H)} c_f}{|V(H)| - 1} \right\}, \quad (5)
$$

where for a vertex $u$, $\delta(u)$ is the set of edges with $u$ as an end-point. It now follows that a graph $G$ will have the desired fractional edge $\Delta$-colouring precisely when there exists an $\epsilon > 0$ such that if we let $e$ have edge weight $c_e = 1 + \epsilon$ and all other edges $f \neq e$ have edge weight $c_f = 1$ then the maximum to Eq. (5) is $\Delta$. To ensure $\max_{u \in V} \sum_{e \in \delta(u)} c_e \leq \Delta$, it is enough to ensure that $\epsilon \leq 1$ and that $|\delta(y)| < \Delta$ and $|\delta(z)| < \Delta$, where $e = yz$. To ensure that $\max_{H \subseteq G} \sum_{f \in E(H)} c_f/|V(H)| - 1 \leq \Delta$, we ensure the more stringent condition that $\max_{H \subseteq G} 2 \sum_{f \in E(H)} c_f/|V(H)| - 1 \leq \Delta$ holds when $c_e = 1 + \epsilon$. To do so, we define an induced subgraph $F$ of $G$ to be $\Delta$-full if

$$
|E(F)| = \Delta \frac{|V(F)| - 1}{2}. \quad (6)
$$

Now using the edge weights described above, if $e$ is contained in some $\Delta$-full subgraph then for any $\epsilon > 0$ Eq. (5) will be greater than $\Delta$. It turns out that if $e$ is not contained in any $\Delta$-full subgraph then by picking $\epsilon$ small enough we can always satisfy the more stringent condition.

We are now in a position to explain the particular vertex colour that we use to prove Lemma 2. Consider a vertex $(\Delta + 2)$-colouring of a graph $G$ with colour classes are $S_1, S_2, \ldots, S_{\Delta+2}$. Let $H_i = G - S_i$ for each $i$ between 1 and $\Delta + 2$. For an edge $e = yz \in E$, $\{S_1, S_2, \ldots, S_{\Delta+2}\}$ is called a \textit{better colouring} for $e$ if

1. $e \in E(H_1)$, $|\delta_{H_1}(y)| < \Delta$, and $|\delta_{H_1}(z)| < \Delta$,
2. $H_1$ has no $\Delta$-full subgraph of $G$ which contains both $y$ and $z$, and
3. for each overfull subgraph $O$ of $G$, $V(O) \cap S_i \neq \emptyset$ for $1 \leq i \leq \Delta + 1$ and $V(O) \cap S_{\Delta+2} \neq \emptyset$ whenever $|V(O)| \geq \Delta + 2$.

The following lemma shows that a better colouring for each edge $e \in E$ is exactly the colouring we seek.
Lemma 3. If $G = (V, E)$ has a better colouring for an edge $e \in E$ then $G$ has a fractional total $(\Delta + 2)$-colouring $w^e$ such that $\sum_{T \ni e} w_T^e > 1$.

Finally, to complete the proof we show

Lemma 4. If $G = (V, E)$ satisfies $\Delta \geq 3$ and $G \neq K_{2n}$ nor $G \neq K_{n,n}$ for some integer $n \geq 1$, then for each edge $e \in E$ there exists a better colour for $e$.

References


