On the Optimality of Multi-Hop Communication in Large Wireless Networks

Urs Niesen
Alcatel-Lucent Bell Laboratories
Murray Hill, NJ 07974
urs.niesen@alcatel-lucent.com

Piyush Gupta
Alcatel-Lucent Bell Laboratories
Murray Hill, NJ 07974
pgupta@research.bell-labs.com

David Tse
University of California at Berkeley
Berkeley, CA 94720
dtse@eecs.berkeley.edu

Abstract—We consider arbitrary traffic patterns in arbitrarily placed extended wireless networks. We provide sufficient conditions for the approximate optimality of multi-hop communication over such networks. For exponential power decay, we show that these sufficient conditions are always satisfied, resulting in a scaling characterization of the entire capacity region for any node placement.

I. INTRODUCTION

The holy grail of network information theory is to characterize the capacity region of a general communication network. While finding such a characterization has proved elusive, progress has been made in recent years to approximate this capacity region. However, these approximation results are usually at the expense of generality in the network considered. In particular, for wireless networks, approximation results have been derived for randomly placed nodes with essentially uniform traffic pattern. This limits the applicability of these approximation results.

In this paper, we consider general extended wireless networks with nodes placed in an arbitrary (i.e., deterministic) manner and with arbitrary traffic pattern. We derive sufficient conditions for the optimality of multi-hop communication in such general wireless networks. For node placements satisfying these conditions, we provide a scaling characterization of the entire capacity region. We show that for exponential power decay, these sufficient conditions are always satisfied. Hence under exponential power decay, we obtain an approximation of the capacity region for arbitrary node placement.

A. Related Work

Consider an extended wireless network in which \( n \) nodes are placed on the square region \([0, \sqrt{n}]^2\) of area \( n \) and communicate with each other over Gaussian fading channels with path-loss exponent \( \alpha \geq 2 \). The object of interest in this paper is the capacity region \( \Lambda(n) \subset \mathbb{R}^{n \times n}_+ \), describing achievable rates between all possible source-destination pairs.

The study of capacity scaling laws was initiated by Gupta and Kumar in [1]. Under a protocol channel model, in which interference is treated as noise and only point-to-point communication is allowed, and under random source-destination pairing with equal traffic demands, the largest achievable per-node rate is shown to scale like \( n^{-1/2 \pm o(1)} \). Hence, under the protocol channel model, and assuming random node placement, [1] provides the scaling behavior of one point of the \( n \times n \)-dimensional capacity region \( \Lambda(n) \).

Subsequent work in the information-theory literature focused on removing the protocol channel model assumption made in [1] and instead considered Gaussian fading channels. In a series of papers [2]–[9], upper bounds on the achievable rates for random source-destination pairing were derived. In particular, it is shown in [8] that for \( \alpha \geq 3 \) multi-hop communication is indeed optimal, and hence the largest uniformly achievable per-node rate scales like \( n^{-1/2 \pm o(1)} \). On the other hand, in another stream of work [8], [10]–[13], it is shown that for \( 2 \leq \alpha < 3 \) cooperative communication schemes significantly outperform multi-hop communication. In particular, [8] introduced a hierarchical cooperative communication scheme achieving the order optimal per-node rate scaling of \( n^{1-\alpha/2 \pm o(1)} \). This provides scaling information, now under the Gaussian fading channel model, but still assuming random node placement, again about one point in \( \Lambda(n) \).

The impact of the random node placement assumption on achievable rates was investigated in [14] where it is shown that for low path-loss exponent \( \alpha < 3 \), the same uniform per-node rate of \( n^{1-\alpha/2 \pm o(1)} \) is achievable regardless of the node placement. On the other hand, in the high path-loss regime \( \alpha \geq 3 \), the regularity of the node placement crucially affects achievable rates as well as the nature of order optimal communication schemes. In particular, it is shown in [14] that there are node placements (containing “gaps”) for which multi-hop communication is not order optimal for any value of \( \alpha \), contrasting with the results for random node placement where multi-hop communication is order optimal for all \( \alpha \geq 3 \).

General traffic patterns were considered in [15] under random node placement. For \( \alpha > 5 \), a scaling characterization of the entire \( n \times n \)-dimensional capacity region \( \Lambda(n) \) is provided. For \( 2 \leq \alpha \leq 5 \) the scaling of all but \( n \) dimensions of \( \Lambda(n) \) is characterized.

If arbitrary node placement as well as arbitrary traffic patterns are allowed, the problem becomes considerably harder to deal with. To the best of our knowledge, no scaling results for this general wireless network setting are known.
B. Organization

The remainder of this paper is organized as follows. In Section II, we introduce network and channel models. We present the main results of this paper in Section III. The communication schemes used to prove the inner bound on the capacity region is described in Section IV. We illustrate applications of the main results in Section V with several examples. Section VI contains concluding remarks.

Due to space constraints, results are presented without their proofs.

II. Models and Notation

We now introduce the network model (Section II-A) and the traffic model (Section II-B).

A. Network Model

Let \( A(n) = [0, \sqrt{n}]^2 \) be a square of area \( n \), and consider an extended network consisting of \( n \) nodes \( V(n) \subset A(n) \) on \( A(n) \) (with \( |V(n)| = n \)). We assume that these \( n \) nodes are placed in an arbitrary (deterministic) fashion on the square region \( A(n) \), i.e., we make no probabilistic assumption on the node placement. Let \( r_{u,v} \) be the Euclidean distance between the nodes \( u \) and \( v \), and denote by \( r_{min}(n) = \min_{u,v \in V(n)} r_{u,v} \) the minimum separation between nodes in \( V(n) \). While the results presented in this paper are valid for all values of \( r_{min}(n) \), the scenario we have in mind throughout is \( r_{min}(n) \geq n^{-o(1)} \).

The nodes \( V(n) \) communicate with each other over the wireless channel modeled as follows. The received signal \( y_{u}[t] \) at time \( t \) at node \( v \in V(n) \) is given by

\[
y_{u}[t] = \sum_{u \neq v} h_{u,v}[t] x_{u}[t] + z_{u}[t],
\]

where \( x_{u}[t] \) is the transmitted signal at node \( u \), \( h_{u,v}[t] \) is the channel gain between nodes \( u \) and \( v \), and \( z_{u}[t] \) is additive noise at node \( v \), all at time \( t \). We impose a unit average power constraint for the transmitted signal \( \{x_{u}[t]\}_{t} \) at every node \( u \in V(n) \). We assume that the noise terms \( \{z_{u}[t]\}_{u} \) are independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with mean \( 0 \) and variance \( 1 \), and independent of the signals \( \{x_{u}[t]\}_{u} \), and the channel gains \( \{h_{u,v}[t]\}_{u,v} \). We further assume that \( \{z_{u}[t]\}_{t} \) are i.i.d. as a function of time \( t \). The channel gains \( \{h_{u,v}[t]\}_{u,v} \) are assumed to have the following structure.

\[
h_{u,v}[t] = r_{u,v}^{-\alpha/2} \exp \left( \sqrt{-1} \theta_{u,v}[t] - \tau r_{u,v} \right),
\]

where \( \{\theta_{u,v}[t]\}_{u,v} \) are random variables taking values between \([0, 2\pi)\). The parameters \( \alpha > 2 \) and \( \gamma \geq 0 \) determine the decay of power as a function of distance. Over distance \( r \), power decays polynomially in \( r \) with exponent \( \alpha \) and exponentially in \( r \) with exponent \( \gamma \). We make no further assumption on the phase terms \( \{\theta_{u,v}[t]\}_{u,v} \), in particular, the phase terms can have arbitrary dependence across nodes \( u \) and \( v \) and time \( t \).

For all the converse results, we assume that full causal channel state information (CSI) is available at all nodes, i.e., each node knows all \( \{\theta_{u,v}[t]\}_{u,v} \) at time \( t \). For achievability results, we assume that only causal local receiver CSI is available, i.e., each node \( v \) knows \( \{\theta_{u,v}[t]\}_{u} \) at time \( t \). The assumption of local receiver CSI is adopted here for ease of exposition, similar achievability results can also be derived assuming no CSI is available at any of the nodes in the network. Making these different assumptions for CSI for converse and achievability results implies that all the bounds presented in this paper are valid under any other assumption on the availability of CSI in between these two cases as well.

B. Traffic Model

A traffic matrix \( \lambda \in \mathbb{R}^{n \times n} \) assigns to each node pair \((u, w) \in V(n) \times V(n)\) the rate \( \lambda_{u,w} \), at which node \( u \) generates traffic for node \( w \). The messages for distinct \((u, w)\) pairs are assumed to be independent, in other words, each message is to be sent to only one destination node. However, we allow the same node \( u \) to be source for several messages (for distinct destination nodes \( v_{1}, v_{2}, \ldots \)), and we allow the same node \( w \) to be destination node for several messages (from distinct source nodes \( u_{1}, u_{2}, \ldots \)). The capacity region \( \Lambda(n) \subset \mathbb{R}^{+ \times n} \) is the collection of all achievable traffic matrices \( \lambda \in \mathbb{R}^{+ \times n} \). Thus, the capacity region determines achievability of every possible traffic pattern.

III. Main Results

We now present the main results of this paper. We provide sufficient conditions for the optimality of multi-hop communication for any traffic matrix. In case these conditions are satisfied, we provide a scaling approximation of the entire \( n \times n \)-dimensional capacity region \( \Lambda(n) \).

For \( d(n) \geq 0 \), define

\[
\bar{\Lambda}(n) \triangleq \left\{ \lambda \in \mathbb{R}^{n \times n} : \sum_{u \in S} \sum_{w \notin S} \lambda_{u,w} \leq \sum_{u \in S} \sum_{v \notin S} \sum_{r_{u,v} \leq d(n)} |h_{u,v}|^2 \text{ } \forall S \subset V(n) \right\}.
\]

\( \bar{\Lambda}(n) \) is the collection of all traffic matrices such that for every cut \( S \subset V(n) \) in the network the total traffic across the cut is less than the sum of squared channel gains between nodes close to the cut (i.e., within distance \( d(n) \)). While \( d(n) \) is allowed to be arbitrary, the situation of interest is \( d(n) \leq n^{o(1)} \).

Call a node placement \((b_{1}(n), b_{2}(n), d(n))\) locally dominated if for every \( S \subset V(n) \)

\[
\sum_{u \in S} \left( \sum_{v \notin S} |h_{u,v}| \right)^2 \leq b_{1}(n) \sum_{u \in S} \sum_{v \notin S} |h_{u,v}|^2 + b_{2}(n).
\]

Intuitively, for small \( b_{1}(n), b_{2}(n), d(n) \), a node placement is locally dominated if across every cut \( S \subset V(n) \) the total power
transfer achievable with beamforming
\[
\sum_{v \in S} \left( \sum_{u \in S} |h_{u,v}| \right)^2
\]
is not much larger than the power transfer for independent signaling between nodes close to the cut
\[
\sum_{u \in S, v \notin S, r_{u,v} \leq d(n)} |h_{u,v}|^2.
\]

The next theorem shows that \( \lambda(n) \) is always approximately achievable using multi-hop communication between nodes of distance at most \( d(n) \). In other words, \( \lambda(n) \) is an approximate inner bound for the capacity region \( \Lambda(n) \). Moreover, if a node placement \( V(n) \) is locally dominated with small enough \( b_1(n), b_2(n) \) then \( \lambda(n) \) is also an approximate outer bound for \( \Lambda(n) \).

**Theorem 1.** There exists \( K > 0 \) such that for any \( \alpha > 2 \), \( \gamma \geq 0 \), \( n \in \mathbb{N} \), \( d(n) \geq 0 \), and node placement \( V(n) \),
\[
K \frac{(\alpha - 2)^{2+\alpha}}{2^{2\alpha} \alpha (d(n) + 1)^2 \log(n)} \lambda(n) \subset \Lambda(n).
\]
If, moreover, \( V(n) \) is \((b_1(n), b_2(n), d(n))\) locally dominated then
\[
\lambda(n) \subset b_1(n) \lambda(n) + b_2(n).
\]

For exponential power decay, we show that every node placement is locally dominated with \( d(n) = \log^2(n) \).

**Theorem 2.** Let \( d(n) = \log^2(n) \). Then there exists \( K > 0 \) such that for any \( \alpha > 2 \), \( \gamma > 0 \), and \( n \in \mathbb{N} \), any node placement \( V(n) \) is \((b_1(n), b_2(n), d(n))\) locally dominated with
\[
b_1(n) \triangleq K \frac{\log^{2/\gamma}(n)}{r_{\min}(n)},
\]
\[
b_2(n) \triangleq \exp(K \log(n) + 2\alpha \log(2r_{\min}(n)^{-1}) - \frac{\gamma}{2} \log^2(n)).
\]

Combining Theorems 1 and 2, we see that for exponential power decay (i.e., \( \gamma > 0 \)), \( \lambda(n) \) approximates the capacity region \( \Lambda(n) \) up to a polylogarithmic factor in \( n \) for any node placement \( V(n) \). More precisely, for \( r_{\min}(n) = \Theta(1) \),
\[
\Omega(\log^{-2}(n)) \lambda(n) \subset \Lambda(n) \subset O(\log^2(n)) \lambda(n) + n^{-\omega(1)}.
\]

In other words, we obtain a fairly tight scaling characterization for arbitrary node placement and arbitrary traffic pattern. Moreover, the result shows that for exponential power decay multi-hop communication is order optimal (up to polylogarithmic factor) regardless of node placement and traffic requirement.

The situation is quite different for polynomial power decay (i.e., \( \gamma = 0 \)). In this case, there exist node placements and traffic patterns such that multi-hop communication is not optimal for any value of \( \alpha \), see Example 3 in Section V. (We point out that this contrasts with the situation for random node placement and random source-destination pairing with uniform rate, for which it is shown in [8] that multi-hop communication is order optimal for \( \alpha > 3 \).) Hence the optimality of multi-hop communication for polynomial power decay has to depend on the placement of the nodes. This dependence is captured by the notion of locally dominated as defined in (1).

**IV. Communication Scheme**

This section provides a qualitative description of the communication schemes used to prove achievability (i.e., the inner bound) in Theorem 1. The scheme is based on multi-hop communication.

Pick some node \( u \in V(n) \) in the network and consider a node \( v \in V(n) \) within distance \( d(n) \) of it. The point-to-point capacity from \( u \) to \( v \) (ignoring interference) is given by \( \log(1 + |h_{u,v}|^2) \approx |h_{u,v}|^2 \). Assuming the minimum distance between nodes is of order \( \Theta(1) \), there are at most \( O(d^2(n)) \) nodes within distance \( d(n) \) of \( u \), and by time-sharing between them, \( u \) can communicate simultaneously with all its neighbors up to distance \( d(n) \) at rates \( \Omega\left(\frac{1}{d(n)} |h_{u,v}|^2\right) \). Since we are dealing with an extended network, the interference power from other nodes communicating concurrently turns out to be on the order of the receiver noise, and hence rate \( \Omega\left(\frac{1}{d^2(n)} |h_{u,v}|^2\right) \) is achievable simultaneously between all pairs \((u,v)\) at distance at most \( d(n) \).

Consider now a source-destination pair \((u,w)\) that is at distance strictly larger than \( d(n) \). Communication between \( u \) and \( w \) is performed through multiple hops of length at most \( d(n) \). In other words, we perform multi-hop communication with hop length up to \( d(n) \).

An equivalent formulation of this communication scheme is as follows. Define an (undirected, capacitated) graph \( G := (V_G, E_G) \) with \( V_G = V(n) \) and with \((u,v) \in E_G \) if \( r_{u,v} \leq d(n) \). Assign each edge \((u,v) \in E_G \) a capacity of order \( \Theta\left(\frac{1}{d^2(n)} |h_{u,v}|^2\right) \). Then multi-hop communication over the wireless network as described in the previous paragraph is equivalent to routing over the graph \( G \). Thus, if we denote by \( \Lambda_G(n) \subset \mathbb{R}^n \times \mathbb{R}^n \) the capacity region (under routing) of \( G \), then \( \Lambda_G(n) \subset \Lambda(n) \), i.e., every traffic matrix \( \lambda \in \Lambda_G(n) \) is achievable.

The assignment of routes between source and destination nodes in \( G \) is performed by solving a linear program. Let \( \lambda \in \Lambda_G(n) \), denote by \( P_{u,w} \) the collection of all paths in \( G \) between \( u \) and \( w \). Denote by \( f_{p,u,w} \) the amount of flow routed from \( u \) to \( w \) over path \( p \in P_{u,w} \). Note that \( \{f_{p,u,w}\}_{p,u,w} \) completely specify the routing scheme. To transmit messages according to the traffic matrix \( \lambda \), we find \( \{f_{p,u,w}\}_{p,u,w} \) by solving

\[
\begin{align*}
\max & \quad \phi \\
\text{s.t.} & \quad \sum_{p \in P_{u,w}} f_{p,u,w} \geq \phi \lambda_{u,w} \quad \forall \, u, w, \\
& \quad \sum_{u, w \in V(n)} f_{p,u,w} \leq c_e \quad \forall \, e \in E_G, \\
& \quad f_{p,u,w} \geq 0 \quad \forall \, u, w, p.
\end{align*}
\]

The linear program (2) computes the largest multiple \( \phi \) such that the total flow from \( u \) to \( w \) is at least \( \phi \) times the demand
\( \lambda_{u,w} \), and all capacity constraints are satisfied. It is easy to see that \( \lambda \in \Lambda_G(n) \) if and only if the corresponding optimal value \( \phi^* \) of \( \phi \) is greater than or equal to one. Moreover, \( \phi^* \lambda \) is the largest multiple of \( \lambda \) that can be routed over \( G \).

While the linear program (2) does not have polynomial size (as the number of paths \( P_{u,w} \) could be exponentially big in \( n \)), the problem can nevertheless be solved in polynomial time. This can be seen by rewriting the problem in terms of per-link flow variables. Moreover, (2) can be solved in a distributed manner (see, for example, [16] for an overview of such distributed algorithms for the routing problem).

We point out that solving a linear program as (2) is indeed necessary for order optimal communication. In Section V, we provide several examples illustrating that simpler routing strategies such as straight-line routing are not sufficient for order optimality.

To conclude the proof of the inner bound, the region \( \Lambda_G(n) \) needs to be linked to the region \( \hat{\Lambda}(n) \) in the statement of Theorem 1. More precisely, we need to argue that \( \Omega(\log^{-1}(n)) \hat{\Lambda}(n) \subset \Lambda_G(n) \). The definition of \( \hat{\Lambda}(n) \) is given in terms of cut-set bounds for the graph \( G \). Hence, to conclude the proof, we need an approximate max-flow min-cut theorem for routing of arbitrary traffic over graphs. A result by Linial, London, and Rabinovich [17] shows that such an approximate max-flow min-cut theorem does indeed hold, and hence yields the inner bound in Theorem 1.

V. Examples

We now present several example scenarios illustrating applications of the main results. Examples 1 and 2 demonstrate the suboptimality of nearest-neighbor multi-hop communication and straight-line (or geographic) routing. Instead multi-hop routes have to be determined taking the entire geography of the node placement into account. This is done by solving a linear program, as is explained in Section IV. Example 3 illustrates that multi-hop communication may not be order optimal for value of \( \alpha \) as long as \( \gamma = 0 \), i.e., depending on the node placement multi-hop communication may only be optimal for exponential power decay.

Example 1. (Suboptimality of nearest-neighbor multi-hop communication)

Let \( \gamma > 0 \) (i.e., exponential power decay). This example illustrates that nearest-neighbor multi-hop communication can be arbitrarily suboptimal. Instead, all hops with length up to at least \( \omega(\log(n)) \) need to be considered.

Consider the node placement in Figure 1. The shaded regions contain \( \Theta(n) \) nodes each, with nodes placed regularly. Between the two shaded regions is a gap of length \( \frac{1}{\gamma^2} \log(n) \), and there is one line of \( \frac{1}{\gamma^2} \log(n) \) nodes connecting them.

If we perform nearest neighbor routing, all communication from the left side of the network to its right side has to be routed through this one line connecting them. This upper bounds the sum rate across this cut using nearest-neighbor multi-hop communication by \( O(1) \). On the other hand, if each node on the boundary of two clusters communicates directly to the closest node in the other cluster, we obtain a sum rate of at least

\[
\Omega\left(\log^{-5}(n)\sqrt{n}\left(\frac{1}{\gamma^2} \log(n)\right)^{-\alpha} \exp\left(-\gamma \frac{1}{4\gamma^2} \log(n)\right)\right)
\]

\[
= \Omega(\log^{-5-\alpha}(n)n^{1/4}) \gg O(1),
\]

across the same cut. Moreover, Theorems 1 and 2 show that no scheme can obtain a sum rate of more than

\[
O(\log^{8-\alpha}(n)n^{1/4})
\]

across this cut. Hence multi-hop communication across this cut with hop-size \( \frac{1}{\gamma^2} \log(n) \) is approximately optimal up to a polylogarithmic factor in \( n \).

We point out that the hop size can not always be taken to be logarithmic. In fact, for regular node placement, the hop size needs to be of order \( \Theta(1) \), and using only logarithmic hop size will result in suboptimal scaling of achievable rates. Thus the hop size cannot be determined a priori, but depends on the node placement and traffic pattern, and needs to be determined by solving the linear program (2).

Example 2. (Suboptimality of straight-line and geographic routing)

Let \( \gamma > 0 \) (i.e., exponential power decay). This example illustrates the suboptimality of straight-line and geographic routing. Instead, the optimal routing scheme has to be solved by taking the entire geography of the node placement into consideration.

Consider the node placement in Figure 2. The two shaded regions contain \( \Theta(n) \) uniformly placed nodes each. They have a boundary of length \( n/4 \) at distance \( \log^2(n) \), and are connected by a line of length \( n/2 \) consisting of \( n/2 \) nodes.

Consider the nodes \( u \) and \( v \) in the figure. Either straight-line or geographic routing between these two nodes has to cross the gap of width \( \log^3(n) \), and can hence achieve at most a rate of

\[
O\left(n \exp\left(-\gamma \log^3(n)\right)\right)
\]
between $u$ and $v$. On the other hand, multi-hop routing along the line of length $n/2$ consisting of $n/2$ nodes achieves a rate of at least
\[ \Omega(1) \gg O\left(n \exp(-\gamma \log^2(n))\right) \]
between $u$ and $v$.

**Example 3.** (Optimality of multi-hop communication for arbitrary node placement)
Consider the node placement consisting of half the nodes placed uniformly on $[0, \sqrt{n}/4] \times [0, \sqrt{n}]$ and the other half placed uniformly on $[3\sqrt{n}/4, \sqrt{n}] \times [0, \sqrt{n}]$. Construct a traffic matrix $\lambda$ by pairing each node in the left rectangle with one randomly chosen node from the right rectangle with uniform rate requirement. For this traffic matrix $\lambda$, it can be shown that multi-hop communication is not order optimal for any value of $\alpha$ whenever $\gamma = 0$. In other words, multi-hop communication is never order optimal for polynomial power decay. On the other hand, Theorem 2 shows that for exponential power decay (i.e., $\gamma > 0$), multi-hop communication is order optimal; however, for the traffic matrix $\lambda$ under consideration, the achievable rate will be equal to $n^{-\omega(1)}$, i.e., the scaling exponent is $-\infty$.

**VI. CONCLUSION**
We considered extended wireless networks with arbitrary node placement. We derived sufficient conditions for the order optimality of multi-hop communication for all traffic patterns. We showed that these conditions are satisfied for exponential power decay regardless of the node placement. This results in a scaling characterization of the entire $n \times n$-dimensional capacity region of arbitrary wireless networks with exponential power decay.

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